ANALYTIC
GEOMETRY

NEWCOMB'S
MATHEMATICAL
COURSE

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ELEMENTS

OF

ANALYTIC GEOMETRY

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PREFACE.

THE author has endeavored so to arrange the present work that it shall be adapted both to those who do and those who do not desire to make a special study of advanced mathematics. Believing it better that a student should learn a little thoroughly and understandingly than that he should go over many subjects without mastering them, the work is so constructed as to offer a wide range of choice in the course to be selected.

The opening chapter contains a summary of the new ideas associated with the use of algebraic language, which the student is now first to encounter. His subsequent progress will depend very largely on the ease and thoroughness with which he can master this chapter.

The next seven chapters correspond closely to the usual college course in plane analytic geometry; but the second sections of Chapters III. and IV., as well as some sections of Chapter VIII., may be regarded as extras in this course.

If to this be added the part on geometry of three dimensions, we shall have a course for those who expect to apply the subject to practical problems in engineering and mechanics.

The second sections of Chapters III. and IV., together with Part III., form an introduction to the modern projective geometry; a subject whose elegance especially commends it to the student of mathematical taste. The author has tried to develop it in so elementary a way that it shall offer no difficulty to a student who has been able to master elementary geometry and trigonometry.

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PLANE ANALYTIC GEOMETRY.

CHAPTER I.

FUNDAMENTAL CONCEPTIONS IN ALGEBRA AND GEOMETRY.

1. Analytic Geometry is a branch of mathematics in which position is defined by means of algebraic quantities.

As an example of how position may be defined by quantities we may take latitude and longitude. The statement

"This ship is in lat. 47° N. and long. 52° W."

indicates to the expert a certain definite point on the earth's surface near Newfoundland.

47° and 52° are the *quantities* indicating the position.

Algebraic Conceptions.

The following algebraic conceptions and principles should be well understood by the student of Analytic Geometry.

2. *Principles of Algebraic Language.*

I. When an algebraic symbol is used in a statement, the statement is considered true for any value of that symbol, unless some limitation is placed upon it.

II. Any algebraic expression represents a quantity, and may itself be represented by a single letter.

3. *Constants and Variables.* A quantity is called

Constant when a definite fixed value is supposed to be assigned it;

Variable when, no definite value being assigned it, it is subject to change.

We may, when we please, assign a definite value to a variable. It then becomes, for the time being, a constant.

EXAMPLE. We may regard the expression

$$x^3 - a^2x$$

as one in which x may take all possible values, while a remains constant: x is then a variable.

But we may also inquire what definite value x must have in order that the expression may vanish. We readily find these values to be

$$x = 0; \qquad x = a; \qquad x = -a.$$

The quantity x then becomes a constant.

Again, we may think of a constant as undergoing variation: it then becomes a variable.

REMARK. The distinction of constant from variable is not an absolute but only a relative one; that is, relative to other quantities, or to our way of thinking at the moment. The only absolute constants are arithmetical numbers.

4. *Functions.* When two variables are so related that a change in one produces a change in the other, the latter is called a **function** of the other.

Such relations between quantities are expressed by algebraic equations.

EXAMPLE. In the equation

$$y = ax + b,$$

a change in a, b or x produces a change in y. Hence y is a function of these quantities.

The value of an algebraic expression containing any symbol will generally vary with that symbol. Hence

Any algebraic expression containing a variable is a function of that variable.

Independent Variables. When a quantity, y, is a function of another quantity, x, we may assign to x all possible values, and study the corresponding values of y.

The quantity x is then called an **independent variable.**

5. *Identical and Conditional Equations.* An equation between algebraic symbols may be either

necessarily true, whatever values be assigned the symbols;

or true only when some relation exists between those values.

An equation necessarily true is called an **identical equation**, or an **identity.** An equation only conditionally true is called an **equation of condition**, or a **condition.**

Between the two members of an identity the sign $\equiv$ is used; between those of a condition, the sign $=$.

EXAMPLE. We have

$$(x + a)(x - a) + a^2 \equiv x^2;$$

because the two members are necessarily equal for all values of x and a. But the statement

$$ax - by = 0$$

can be true only when

$$x = \frac{b}{a} y,$$

and is therefore a *condition.*

The question whether an equation is an identity or a condition is settled by reducing or solving it.

If an *identity,* the two members may be reduced to the same expressions, or, if we try to solve it, we shall only bring out $0 = 0$.

If a *condition,* a value of x in terms of the remaining quantities will be possible.

THEOREM. *An equation of condition becomes an identity by solving it with respect to any one symbol, and substituting the value of the symbol thus found in the equation.*

EXAMPLE. If in the preceding equation

$$ax - by = 0$$

we substitute the value of x derived from it, we have

$$a\frac{b}{a}y - by \equiv 0,$$

an identity.

Hence *any equation may be regarded as an identity by supposing any one of its symbols to represent that function of its other quantities obtained by solving it.*

EXAMPLE. The equation

$$aP - by = 0$$

changes into the identity

$$aP - by \equiv 0$$

when we suppose $$P \equiv \frac{b}{a}y.$$

6. The symbol $\equiv$ is also used as the symbol of definition when, in accordance with § 2, II., we use a symbol to represent an expression. For example,

$$ax + by \equiv X$$

means,

"we use X for brevity, to represent the expression $ax + by$."

When the sign $\equiv$ follows an expression in this way, it may be read, "*which let us call.*"

7. LEMMA. *Between the variables x and y and the constants A, B and C the identity*

$$Ax + By + C \equiv 0 \tag{1}$$

subsists when, and only when,

$$A = 0, \qquad B = 0, \qquad C = 0. \tag{2}$$

Proof. That the identity subsists in the case supposed is obvious; that it subsists only in this case is seen by showing, first, that if C were different from zero, the identity would fail for $x = 0$, $y = 0$; and next, that, C being zero, the identity would fail for $x = 0$ when B was finite, and for $y = 0$ when A was finite.

REMARK 1. The deduction of (2) from (1) rests on the assumption that x and y are independent variables. If A and B were regarded as variables, the conclusions would be

$$x = 0; \qquad y = 0; \qquad C = 0.$$

REMARK 2. Note the great difference between the interpretation of the equation

$$ax + by + c = 0$$

and of the identity

$$ax + by + c \equiv 0.$$

The *equation* expresses a certain relation between the variables x and y such that to each definite value of x corresponds a definite value of y, namely, the value

$$y = -\frac{ax + c}{b}.$$

The *identity* expresses no relation between the quantities, but requires zero values of a, b and c.

8. *Roots of Quadratic Equations.* Every quadratic equation is considered to have two roots, which may be *real* and *unequal, real* and *equal,* or *imaginary.* If the equation is

$$ax^2 + bx + c = 0, \tag{a}$$

then, since we know the roots to be given by the equation

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2c}{-b \mp \sqrt{b^2 - 4ac}},$$

we see that the roots will be

real when $b^2 - 4ac > 0$, i.e., when $b^2 - 4ac$ is positive;

real and *equal* when $b^2 - 4ac = 0$;

imaginary when $b^2 - 4ac < 0$; i.e., when $b^2 - 4ac$ is negative.

The student should now be able to explain the following special cases:

1. If the absolute term c vanishes, the roots become

$$x = 0 \qquad \text{and} \qquad x = -\frac{b}{a}.$$

2. If b and c both vanish, both roots become zero.

3. If a approaches zero as its limit, one root increases without limit, and the other approaches the limit $-\dfrac{c}{b}$.

Hence we may say: When the coefficient of x^2 in the quadratic equation (a) vanishes, the two roots are

$$x = -\frac{c}{b} \qquad \text{and} \qquad x = \infty.$$

4. If both a and b vanish while c remains finite, both roots increase to infinity.

5. If a, b and c all vanish, the roots are entirely indeterminate, and the equation is satisfied by *all values of* x.

9. *Proportional Quantities.* The quantities of one series, a, b, c, etc., are said to be **proportional** to those of another series, A, B, C, etc., when each quantity of the one series is equal to the corresponding quantity of the other multiplied by the same factor.

The fact of such proportionality is expressed in the various forms:

$$a : A = b : B = c : C = \text{etc.};$$
$$a : b : c : \text{etc.} = A : B : C : \text{etc.};$$
$$a = \rho A, \quad b = \rho B, \quad c = \rho C, \quad \text{etc.};$$

ρ being, in the last case, the multiplying factor.

TEST EXERCISES.*

1. A point at the distance (1) from one side of a right angle and at the distance (2) from the other side will be at the distance (3) from the vertex of the angle.

Here the student will substitute symbols at pleasure for (1) and (2), and will replace (3) by the proper function of those symbols, reading the statement accordingly. For example, he may put

x in place of (1) and y in place of (2),
$y + x$ in place of (1) and $y - x$ in place of (2),
etc. etc. etc.,

* These exercises are designed to decide the question whether the student has a sufficient command of algebraic language and of geometrical conceptions to enable him to proceed with advantage to the study of Analytic Geometry. If he can perform all the exercises with ease, he is probably well prepared to go on; if he performs them only with difficulty, he may need much assistance in understanding the subject.

and in each case he must read the statement with the proper expression in place of (3).

2. If, in the preceding example, (1) varies while (2) remains constant, the point will describe a line —— to —— — — — —.

Fill the blanks with appropriate words.

3. If (2) varies while (1) remains constant, the point will describe a line —— to the — — — — —.

4. If (1) remains equal to (2), but both vary, the point will move along the —— — — —.

5. If (1) and (2) vary in such a way that (3) remains constant, the point will describe a —— of radius —— around — — — — — as a —.

6. If two fixed points, A and B, are at the distance a from each other, and if a third point, P, be taken at the distance (4) from each of these points; then, if (4) varies, the point P will describe a line (*define the situation of the line*). But, in varying, the distance (4) cannot become less than —.

The numbers in parentheses are to be replaced by appropriate symbols or expressions.

7. If, in the preceding example, a point be taken at the distance (5) from the point A, and at the distance (6) from B; then, if (5) varies while (6) remains constant, the point will describe a —— of —— — around — — — as a —.

8. But if (6) varies while (5) remains constant, the point will . . . .

9. If the constant value of (6) in Ex. 7 *plus* the constant of (5) in Ex. 8 = a, the two —— will be —— to each other.

10. If a line be drawn so as to pass at the distance r from each of the preceding points, then, if r varies, the line will turn round (*describe how it will turn*). But the value of r can never exceed ——, and for each value of r there will be two positions of the line making equal angles with —.

11. If the line be required to pass at the distance (7) from the point A, and at the distance (8) from the point B; then, if (7) varies while (8) remains constant, the line will move

round so as always to be tangent to the —— around — as a —— with radius —.

12. What symbols must (9) and (10) be replaced by in order that all values of x and y which satisfy the equation

$$Ax + By + C = 0$$

may also satisfy the equation

$$mAx + (9)y + (10) = 0;$$

that is, in order that these two equations may give the same value of y in terms of x?

13. Show that the identity

$$ax + by + c \equiv Ax + By + C,$$

x and y being variables, is impossible unless

$$a = A; \qquad b = B; \qquad c = C.$$

14. If we put

$$P \equiv x - 2y + 3c, \qquad P' \equiv 3x - 6y + 9c,$$

is it possible to form an identity of the form

$$P + mP' \equiv 0,$$

and, if so, what will be the value of m?

15. Generalize the preceding result by showing that if we have

$$P \equiv ax + by + c, \qquad P' \equiv Ax + By + C,$$

the identity $\qquad P + mP' \equiv 0,$

x and y being variables, is possible only when

$$a : A = b : B = c : C,$$

and express the value of m.

16. If $a + x$ remains constant, and x varies at the rate of plus one foot a second, at what rate will a vary?

17. What will be the answer to this last example if it is $a - 2x$ instead of $a + x$ which remains constant?

18. If x may take any values between the extremes -1 and $+2$, between what extremes will the value of $\dfrac{x}{x-1}$ be contained?

Geometric Conceptions.

10. A geometric concept, form, or figure of any kind, may be called a **geometric object.***

In the higher geometry all geometric objects, when not qualified, are considered as complete in every particular.

EXAMPLES. A straight line is considered to extend to infinity in both directions. When a terminating straight line is treated, it is considered as that portion of an infinite straight line contained between some two points.

A triangle is considered as formed by three indefinite straight lines intersecting each other in three different points.

Geometric objects differ from each other in *magnitude*, *form* and *situation*.

Points, straight lines and planes can, however, differ only in situation, because any two points, any two lines or any two planes may be made to coincide with each other by a change of situation.

11. *Points at Infinity.* A pair of parallel lines are said to intersect in a *point at infinity;* that is, in a point at an infinite distance.

The idea of a point at infinity is reached in this way: Let us suppose one of two intersecting lines to turn round on one of its points and gradually to approach the position of parallelism to the other line. As this position is approached the point of intersection of the two lines will recede indefinitely, in such wise that while the revolving line approaches parallelism as its limit, the point will recede beyond every assignable limit.

Conversely, if we suppose the point of intersection to recede indefinitely along the fixed line, the moving line will approach

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* This is the best English word which has presented itself to the author to correspond to the *Gebild* of the Germans. Such a word is needed in the higher geometry as a term of the most general kind to express the things reasoned about. The term *magnitude* is too limited, not only because a point is to be included among geometric objects, but because objects are considered not merely as magnitudes but, in a more general way, as things of which magnitude is only one of the qualities.

the position of parallelism as its limit. This limit will be the same whether the point of intersection recedes in one direction or in the opposite.

Hence, using the convenient language of infinity, we see that when the point of intersection is at infinity in either direction the two lines are parallel. There is, therefore, no need of making any distinction between these supposed points, and they are talked about as a single point, called the point at infinity.

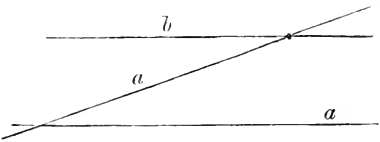
The principle here involved is of extensive application in the higher mathematics, and may be expressed thus:

Instead of using new or different forms of language to meet exceptional cases, we use the common language, but put an exceptional interpretation upon it.

The advantage of this way of speaking is that we are not obliged to make any exceptional cases respecting the intersection of lines when the two lines become parallel.

The proposition, *Two straight lines intersect in a single point*, is then considered universally true, the point being at infinity when the lines are parallel.

The following is a convenient illustration of this form of language. Let it be required to draw a line through a fixed point, P, so as to intersect the fixed line b at the same point, Q, where the line a intersects it. The construction will be literally possible so long as a and b intersect, but will cease to be *literally* possible if a takes the position a', parallel to b, because then there will be no point Q of intersection.



But let us interpret the problem in this way: The required line must intersect b where a intersects b. But in case of parallelism, a intersects b *nowhere*. Hence the required line must intersect b *nowhere;* that is, it must be parallel to it. It is this particular *nowhere* which is called *the point at infinity* on the line b. It is, moreover, clear that if Q recedes to infinity, both the line a and the required line will approach the position of parallelism to b as their respective limits.

CHAPTER II.

OF CO-ORDINATES AND LOCI.

12. *Def.* The **co-ordinates** of a geometric object are those *quantities* which determine its situation.

Co-ordinates, like other quantities, are represented by numerical or algebraic symbols.

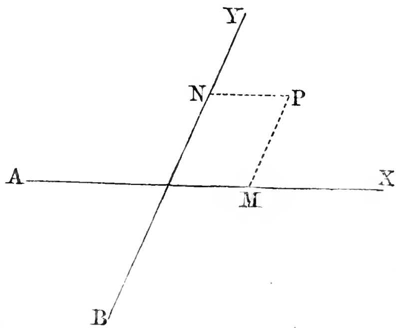
The situation of an object is defined by its relations to some system of points or lines supposed to be fixed. Such a system is called a **system of co-ordinates.**

There are several systems of co-ordinates to be separately defined.

First System: Cartesian or Bilinear Co-ordinates.

13. On this system the position of a point is fixed by its relation to two intersecting straight lines called **axes.**

Let AX and BY be the two lines, and O their point of intersection.



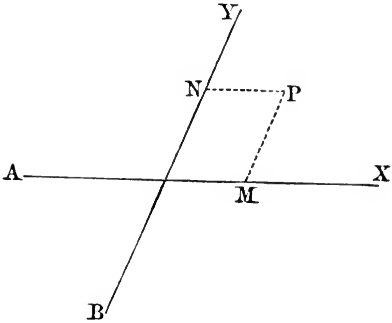
The point O is then called the **origin.**

The indefinite line AX, which we may conceive to be horizontal, is called the **axis of abscissas,** or the **axis of** X.

The intersecting line BY is called the **axis of ordinates,** or the **axis of** Y.

Let P be the point whose position is to be defined.

From P draw PM parallel to OY and meeting the axis of X in M, and



PN parallel to OX and meeting the axis of Y in N.

Then either of the equal lines OM, NP is called the **abscissa** of the point P;

Either of the equal lines MP, ON is called the **ordinate** of the point P.

It is evident that for every position we assign to P the abscissa and ordinate will each have a definite value.

14. *Co-ordinates Determine a Point.* When the lengths OM and MP are given, the point P is completely determined in the following way: We measure from O on the axis of X the given distance OM.

Through M we draw an indefinite line parallel to the axis of Y, and on this line measure a length MP.

The single point P which we thus reach is the point which has the given abscissa and ordinate.

Because the abscissa and ordinate thus determine the situation of P, they form, by definition, a pair of *co-ordinates* of P (§ 12).

Notation. The abscissa is represented by the symbol x.
The ordinate is represented by the symbol y.

It is evident that if the point P be fixed in position, its co-ordinates will be constants. But if P varies, one or both of the co-ordinates will vary also.

15. *Algebraic Signs of the Co-ordinates.* In what precedes it is supposed that the direction, as well as the distance, of the measures OM and MP is given. If these directions were arbitrary, we might measure the given distance OM in either direction from O, and thus reach either the point M to the right of O or the point M' to the left of O.

By measuring the ordinate in either direction from the points M and M' we should reach either of four points, P, P', P'', P''', of which the co-ordinates would all be equal in *absolute* value.

To avoid ambiguity in this respect the algebraic sign of the abscissa is supposed

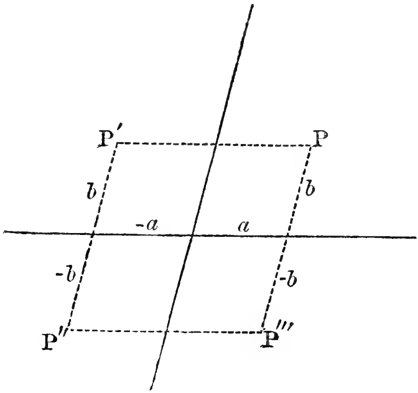
positive when measured from O towards the *right*, and

negative when measured towards the *left*.

The ordinate MP is supposed

positive when measured *upward*, and

negative when measured *downward*.



Now if the abscissa $x = OM = a$ and the ordinate $y = MP = b$, then the

co-ordinates of P are $x = + a, \quad y = + b$;
co-ordinates of P' are $x = - a, \quad y = + b$;
co-ordinates of P'' are $x = - a, \quad y = - b$;
co-ordinates of P''' are $x = + a, \quad y = - b$.

Thus the ambiguity is completely avoided when the algebraic signs of the co-ordinates, as well as their absolute values, are given, so that only one point corresponds to one pair of algebraic values of the co-ordinates.

16. *Rectangular Co-ordinates.* When not otherwise expressed, the axes of co-ordinates are supposed to intersect at right angles. The co-ordinates are then called **rectangular** co-ordinates.

To designate a point by its co-ordinates we enclose the symbols or numbers expressing the co-ordinates between parentheses, with a comma between them, writing the value of x first.

EXAMPLE. By (2, 3) we mean "the point of which the abscissa is 2 and the ordinate is 3."

EXERCISES.

1. Draw a pair of rectangular axes, and, taking a centimetre or inch, as may be most convenient, for the unit, lay down the position of points having the following co-ordinates:

$$(+2, +3),\ (+2, -3),\ (-2, +3),\ (-2, -3),$$
$$(+3, +2),\ (+3, -2),\ (-3, +2),\ (-3, -2).$$

Show that these eight points all lie on a circle having the centre as its origin. What is the radius of this circle?

2. Mark a number of points of each of which the ordinate shall be equal to the abscissa. How are these points situated?

3. Mark the points (1, − 1), (2, − 2), (− 1, 1) and (− 2, 2), and show their relations.

4. Mark the points (1, 2), (2, 4), (3, 6), (4, 8), and show how they are situated relatively to each other.

5. If we join the points $(a, -b)$ and (a, b) by a straight line, what will be the direction of this line?

6. Find, in the same way, the direction of the line joining the points (a, b) and $(-a, b)$; (a, b) and $(-a, -b)$.

7. Show that the distance of the point (a, b) from the origin is $\sqrt{a^2 + b^2}$.

8. If we mark all possible points for which y has the constant value + 1, how will these points be situated?

17. PROBLEM I. *To express the distance between two points whose co-ordinates are given.*

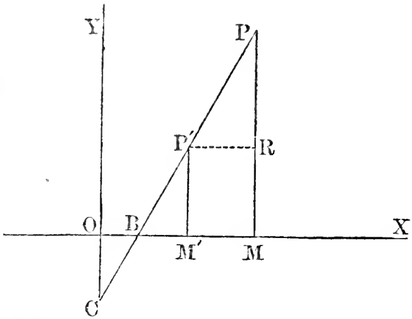
When the co-ordinates of two points are given, the position of each point is completely determined (§ 14).

Therefore the distance between the points is completely determined, and may be measured geometrically.

The algebraic problem requires us to express this distance algebraically in terms of those quantities which determine the position of the points, namely, their co-ordinates.

In the figure let P' and P be the two points; x', y', the co-ordinates of P'; and x, y, the co-ordinates of P.

Then we shall have



$$OM' = x', \quad OM = x;$$
$$P'M' = y', \quad PM = y.$$

If from P' we drop a perpendicular, $P'R$, upon MP, we shall have, from the right-angled triangle $P'PR$,

$$P'R = M'M = x - x',$$

and

$$RP = MP - MR = y - y'.$$

Then, by the Pythagorean proposition,

$$\overline{P'P}^2 = \overline{P'R}^2 + \overline{RP}^2.$$

Let us then put $d \equiv$ the distance $P'P$.

By substituting the values in terms of the co-ordinates and extracting the square root, we shall have

$$d = \sqrt{\{(x - x')^2 + (y - y')^2\}}, \qquad (1)$$

which is the required expression for the distance of the points in terms of their co-ordinates.

18. Problem II. *To express the angle which the line joining two points, given by their co-ordinates, makes with the axis of X.*

Using the same construction as before, let B be the point in which the line PP' intersects the axis of X.

The required angle will then be

$$PBX \quad \text{or} \quad PP'R.$$

If we put

$$\varepsilon \equiv \text{the required angle},$$

we shall have, by trigonometry,

$$RP = P'P \sin \varepsilon = d \sin \varepsilon;$$
$$P'R = P'P \cos \varepsilon = d \cos \varepsilon;$$

whence, by division,

$$\tan \varepsilon = \frac{RP}{P'R} = \frac{y - y'}{x - x'}. \qquad (2)$$

The last equation gives the required expression for the tangent, from which ε may be found.

19. The two preceding problems may be more elegantly solved by a single pair of equations:

$$\left.\begin{aligned} d \sin \varepsilon &= y - y'; \\ d \cos \varepsilon &= x - x'. \end{aligned}\right\} \tag{3}$$

The method of solving these equations is explained in trigonometry.

20. PROBLEM III. *Two points being given by their co-ordinates, it is required to find the points in which the straight line joining them intersects the respective axes of co-ordinates.*

Solution. Let B be the point in which the line intersects the axis of x, C the point in which it intersects the axis of y.

The point B will then be given by the value of OB, its abscissa, which we denote by x_0, and C by the value of OC, its ordinate, which we denote by y_0.

In the similar triangles MBP and $RP'P$ we have

$$BM : MP = P'R : RP.$$

Substituting for the lines their values in terms of the co-ordinates, this gives

$$BM = \frac{y(x - x')}{y - y'};$$

whence

$$OB = OM - BM = x - \frac{y(x - x')}{y - y'}$$

or

$$x_0 = \frac{x(y - y') - y(x - x')}{y - y'} = \frac{xy' - x'y}{y' - y}. \tag{4}$$

The value of OC can be found by a similar construction, but we may also deduce it from OB by the equation

$$OC = OB \tan \varepsilon.$$

But in the figure as drawn C falls below O, so that the value of OC just obtained is the negative of the required

ordinate of the point of intersection. This co-ordinate being y_0, we shall have

$$y_0 = - OB \tan \varepsilon = \frac{xy' - x'y}{x - x'}. \qquad (5)$$

The student should now note the relation between the conditions of the geometric and the algebraic solutions. The problem considered as a geometric one is:

Two points being given in position, to find the intersection of the straight line joining them with the axes of co-ordinates.

The problem is solved geometrically simply by drawing the line. The algebraic requirement is:

Two points being given by means of their co-ordinates, it is required to express the points in which the straight line joining them intersects the co-ordinate axes in terms of the respective co-ordinates of the given points.

The algebraic solution is given by the equations (4) and (5).

21. The preceding problems illustrate the following general principle:

Whenever one geometric object is determined by another geometric object, the algebraic quantities which define the one can be expressed in terms of those quantities which define the other.

EXERCISES.

1. Lay down the four points (1, 1), (1, 2), (2, 2), (2, 1), and join each one and that next following so as to form a quadrilateral. What will be the nature of this quadrilateral?

2. Show that the points (1, 0), (1, 1), (2, 0), (2, 1) lie at the four vertices of a square.

3. Show that each of the following sets of four points are the vertices of a parallelogram:

Set (*a*): (0, 0), (3, 1), (0, 4), (3, − 3).
Set (*b*): (1, 3), (2, 5), (6, 5), (5, 3);
Set (*c*): (1, 1), (2, 4), (5, 5), (4, 2).

4. Show by a geometric construction, employing the properties of similar triangles, that each of the lines joining

the following pairs of points passes through the origin of co-ordinates:

(*a*): a line joining points (1, 1) and (2, 2);
(*b*): a line joining points (1, 2) and (3, 6);
(*c*): a line joining points (1, 3) and (− 1, − 3);
(*d*): a line joining points (a, b) and (na, nb).

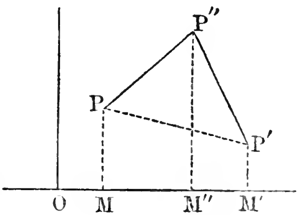
5. Show in the same way that each of the following triplets of points lies in a straight line:

(*a*):	(1, 1),	(2, 2),	(3, 3);
(*b*):	(1, 0),	(2, 2),	(3, 4);
(*c*):	(− 1, 0),	(0, + 2),	(1, + 4);
(*d*):	(3, − 2),	(1, − 1),	(− 1, 0);
(*e*):	(a, b),	$(a + p, b + q)$,	$(a + np, b + nq)$.

6. What are the distance and direction (relatively to the axis of X) from the point (1, 2) to the point (4, 6)?

22. PROBLEM IV. *To find the area of a triangle, the co-ordinates of the vertices being given.*

REMARK. Since the positions of the vertices completely determine the triangle, and therefore determine its area also, it follows from the general principle, § 21, that this area can be algebraically expressed in terms of the co-ordinates of the vertices.



Solution. Let P, P' and P'' be the vertices, and (x, y), (x', y') and (x'', y'') their respective co-ordinates.

Let us put Δ the area of the triangle. We shall then have

$\Delta =$ area $PMM''P''$ *plus* area $M'P'P''M''$ *minus* area $MM'P'P$.

In these three trapezoids we have

$$\text{area } MPP''M'' = \tfrac{1}{2}(MP + M''P'')MM'' = \tfrac{1}{2}(y + y'')(x'' - x);$$

$$\text{area } M''P''P'M' = \tfrac{1}{2}(M''P'' + M'P')M''M' = \tfrac{1}{2}(y'' + y')(x' - x'');$$

$$\text{area } MPP'M' = \tfrac{1}{2}(M'P' + MP)MM' = \tfrac{1}{2}(y' + y)(x' - x).$$

Therefore

$$2\Delta = (y + y'')\,(x'' - x) + (y'' + y')\,(x' - x'') + (y' + y)\,(x - x'),$$

or, by reduction,

$$2\Delta = y(x'' - x') + y'(x - x'') + y''(x' - x), \qquad (6)$$

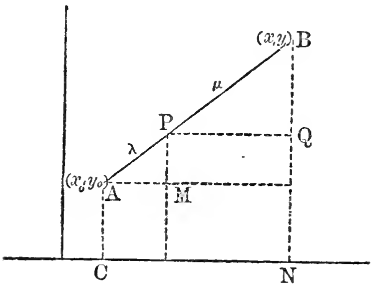
which is the required expression.

23. *To divide a finite line into segments having a given ratio.* A finite line is defined by the co-ordinates of its two terminal points. Let us now consider the problem:

To find the co-ordinates of the point which divides the finite line joining two given points into segments having a given ratio.

Let us put:

x_0, y_0, the co-ordinates of one end, A, of the line.
x_1, y_1, the co-ordinates of the other end, B.
λ, μ, the given ratio.
x, y, the co-ordinates of the required point, P.



Draw AM and PQ each parallel to the axis of X, and PM, BN each parallel to the axis of Y. Then

$AM = x - x_0$; $PQ = x_1 - x$;
$PM = y - y_0$; $BQ = y_1 - y$.

Since we require that

$$AP : PB = \lambda : \mu,$$

we have the proportion

$$\begin{aligned}\lambda : \mu &= AP : PB \\ &= AM : PQ = x - x_0 : x_1 - x \\ &= PM : BQ = y - y_0 : y_1 - y.\end{aligned}$$

We hence deduce the equations

$$\lambda(x_1 - x) = \mu(x - x_0),$$
$$\lambda(y_1 - y) = \mu(y - y_0),$$

which give

$$x = \frac{\lambda x_1 + \mu x_0}{\lambda + \mu}; \qquad y = \frac{\lambda y_1 + \mu y_0}{\lambda + \mu}; \qquad (7)$$

which are the required co-ordinates of the point of division.

Corollary. If P is to be the middle point of the line, we have $\lambda = \mu$, whence

$$x = \frac{x_0 + x_1}{2}; \qquad y = \frac{y_0 + y_1}{2}; \tag{8}$$

or,

Each co-ordinate of the middle point of a line is half the sum of the corresponding co-ordinates of its terminal points.

EXERCISES.

1. Express the co-ordinates of the middle point of the line terminating in the points (1, 6) and (3, − 4).

2. One end of a line is at the point (− 2, − 3) and its middle point at (1, − 2). Where is the other end?

3. Find the middle point of that segment of the line joining the points (− 1, 6) and (3, − 2) which is contained between the axes of co-ordinates. *Ans.* (1, 2).

4. A line terminating at the points (1, 6) and (3, − 4) is to be divided into four equal segments. Find the co-ordinates of the three dividing points.

5. The line joining the points (a, b) and (p, q) is to be divided into *five* equal parts. Express the co-ordinates of the four points of division.

6. What is the distance between the middle points of the lines whose respective termini are in the points (1, 7), (− 5, 3) and (0, 2), (6, − 4)?

7. What point bisects the line from the origin to the middle point of the line terminating at the points (7, − 9) and (− 3, − 7)?

8. Find the co-ordinates of the point which is two thirds of the way from the point (a, b) to the point (a', b').

9. Prove the theorem that the three medial lines of a triangle meet in a point two thirds of the way from each vertex to the opposite side, as follows:

Let (x_0, y_0), (x_1, y_1) and (x_2, y_2) be the three vertices of the triangle.

Express the middle point of each side.

Then express the co-ordinates of those three points which

are respectively two thirds of the way from the several vertices to the middle points of the opposite sides, and thus show that the three points are coincident.

10. Prove that the lines joining the middle points of the opposite sides of a quadrilateral and the line joining the middle points of the diagonals all bisect each other.

To do this, express the co-ordinates of the middle points of the sides and of the diagonals, and then of the middle points of the three joining lines, and show that the latter points are the same for each joining line. The very simple proof of this theorem which is thus found affords a striking example of the power of the analytic method.

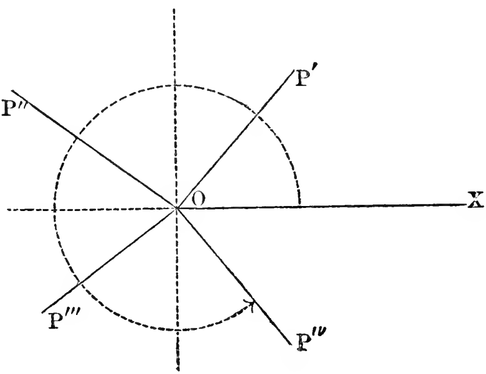
Second System: Polar Co-ordinates.

The position of a point may be defined by its *distance* and *direction* from a fixed point.

The fixed point is then called the **origin.**

The distance of the point from the origin is called the **radius vector** of the point.

In plane geometry the direction of a point from the origin is fixed by the angle which the radius vector makes with an adopted base-line.



Let OX be the base-line, and P the point; P being in any one of the positions P', P'', etc.

OP will then be the radius vector, and the angle XOP will be the required angle.

We generally put

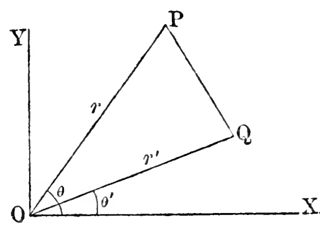
$r \equiv$ the radius vector OP, and

$\theta \equiv$ the angle XOP, which is called the **vectorial angle**.

The former is always considered positive, being measured from the origin, O, in the direction OP. The latter is positive when measured in the direction opposite to that in which the hands of a watch move, and negative in the opposite direction, just as in trigonometry.

24. PROBLEM. *To express the distance between two points in terms of their polar co-ordinates.*

Let P and Q be the points.



In the triangle POQ we have, by trigonometry,

$$PQ^2 = r^2 + r'^2 - 2rr' \cos POQ$$
$$= r^2 + r'^2 - 2rr' \cos(\theta - \theta'),$$

θ and θ' being the angles which the radii vectores make with the initial or base line; therefore

$$PQ = \{r^2 + r'^2 - 2rr' \cos(\theta - \theta')\}^{\frac{1}{2}}, \tag{9}$$

which is the distance required.

EXERCISES.

1. Show how a point will be situated when its vectorial angle is in the first, second, third and fourth quadrant respectively.

2. If the vectorial angle θ' and radii vectores r and r' are constant, while θ may vary at pleasure, for what values of θ will the distance of the points be the greatest and least possible, and what will be the greatest and least distances? Show the correspondence of the algebraic answer from equation (9) with the obvious answer from the figure.

3. If $r = r'$ and $\theta + \theta' = 180°$, show both geometrically and algebraically that distance $= 2r \cos\theta$.

4. If $\theta - \theta' = 90°$ or $270°$, express the distance of the points both by a diagram and by the equation (9).

Transformation of Co-ordinates from One System to Another.

The general problem of the transformation of co-ordinates is this:

GIVEN: 1. *The co-ordinates x and y of a point P referred to some system of co-ordinates.*

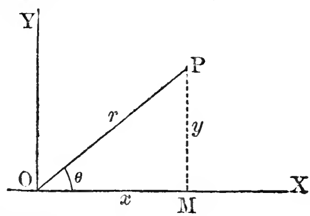
GIVEN: 2. *The position of a second system of co-ordinates in relation to the other system.*

REQUIRED: *To express the co-ordinates of P when referred to the second system.*

25. *Relation of Rectangular and Polar Co-ordinates.*

Let OX, OY be the rectangular axes, and P the position of any point.

1st. We shall suppose the origin to be taken as the pole, and the axis of abscissas as the base or initial line; then we shall evidently have



$$x = r \cos \theta$$

and $$y = r \sin \theta.$$

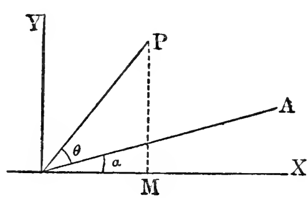
To express the polar co-ordinates in terms of the rectangular co-ordinates, we have from the last two equations, by squaring and adding,

$$r^2 = x^2 + y^2, \qquad \text{or} \qquad r = \sqrt{x^2 + y^2},$$

and, by division, $$\tan \theta = \frac{y}{x},$$

which determine r and θ when x and y are given.

2d. If the initial or base line instead of coinciding with the axis of X makes an angle α with it, we shall evidently have, from the figure,

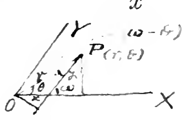


$$x = r \cos (\alpha + \theta)$$

and $$y = r \sin (\alpha + \theta),$$

whence $r = \sqrt{x^2 + y^2}$ and $\tan (\alpha + \theta) = \dfrac{y}{x}.$

3d. From oblique to polar.



$x \sin \omega = r \sin (\omega - \theta)$

$y \sin \omega = r \sin \theta$

EXERCISES.

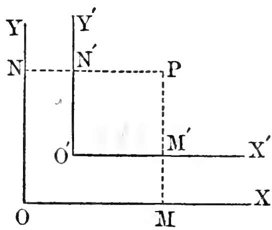
1. In the figure of § 24, express the area of the triangle OPQ in terms of r, r' and $\theta - \theta'$.

2. If r, r' and r'' are the radii vectores, and θ, θ' and θ'' the corresponding angles of three points, which we shall call P, P', P'', it is required to express the areas, first, of the triangles OPP', $OP'P''$ and OPP'', and then of $PP'P''$.

3. The point (3, 3) is the centre of a circle of radius 2, in which two diameters, each making angles of 45° with the axes, are drawn. Find the polar co-ordinates of the ends of these diameters.

4. The point (a, b) is the centre of a circle of radius P. From the centre is drawn a radius making an angle γ with the axis of X. Express the rectangular co-ordinates of the end of this radius.

26. *Transformation from one rectangular system to another.*



Solution. Let us first suppose the two systems of co-ordinates parallel. Also suppose

OX, OY the axes of the original system;

$O'X'$, $O'Y'$ the axes of the second system;

P the point whose co-ordinates are x and y in the old system.

Draw

$$PM'M \parallel YO \parallel Y'O',$$
$$PN'N \parallel XO \parallel X'O',$$

and put

$a \equiv$ the abscissa of the new origin, O', referred to the old system;

$b \equiv$ the ordinate of O';

$x' \equiv$ the abscissa $O'M'$ of P referred to the new system;

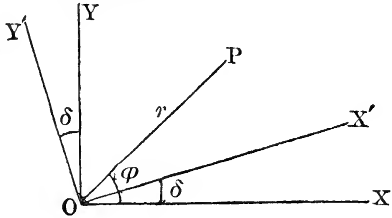
$y' \equiv$ the ordinate $M'P$ of P referred to the new system.

We then have

$$\left.\begin{array}{l} x' = x - a; \\ y' = y - b; \end{array}\right\} \qquad (10)$$

which are the required expressions for the new co-ordinates in terms of the old ones.

27. *Secondly.* Suppose the new axes to make an angle, δ, with the old ones, but to have the same origin.



Let us put

$r \equiv$ the radius vector OP;

$\varphi \equiv$ the angle XOP.

We shall then have

Angle $X'OP = \varphi - \delta$.

Putting, as before, x' and y' for the co-ordinates referred to the new system, and x and y the co-ordinates of the old system, we have, by § 25,

$$x = r\cos\varphi; \qquad y = r\sin\varphi; \qquad (a)$$
$$x' = r\cos(\varphi - \delta); \qquad y' = r\sin(\varphi - \delta). \qquad (b)$$

By trigonometry,

$$\cos(\varphi - \delta) = \cos\varphi\cos\delta + \sin\varphi\sin\delta;$$
$$\sin(\varphi - \delta) = \sin\varphi\cos\delta - \cos\varphi\sin\delta.$$

Substituting these values in (b) and eliminating r and φ by (a), we have

$$\left.\begin{aligned} x' &= y\sin\delta + x\cos\delta; \\ y' &= y\cos\delta - x\sin\delta; \end{aligned}\right\} \qquad (11)$$

which are the required expressions.

To express the old co-ordinates in terms of the new co-ordinates, we have

$$\left.\begin{aligned} x &= x'\cos\delta - y'\sin\delta; \\ y &= x'\sin\delta + y'\cos\delta. \end{aligned}\right\} \qquad (12)$$

If we take for δ the angle which the new axis of Y makes with the old axis of X, the new axis of X will make an angle of $\delta - 90°$ with the old one. Hence in this case the formulæ of transformation will be found by writing $\delta - 90°$ for δ in (12), which gives

$$\left.\begin{aligned} x &= x'\sin\delta + y'\cos\delta; \\ y &= -x'\cos\delta + y'\sin\delta. \end{aligned}\right\} \qquad (13)$$

28. *Thirdly.* Let the new system of co-ordinates have any origin and direction whatever, and let us put, as before,

a, $b \equiv$ the co-ordinates of the new origin referred to the old system;

δ, the angle which each axis of the new system forms with the corresponding axis of the old one.

Imagine through the new origin O' an intermediate system of co-ordinates parallel to the old system, and let us put x_1 and y_1 the co-ordinates of P referred to this intermediate system.

Then, by (10),

$$x_1 = x - a; \qquad y_1 = y - b.$$

By (11),

$$x' = y_1 \sin\delta + x_1 \cos\delta;$$
$$y' = y_1 \cos\delta - x_1 \sin\delta.$$

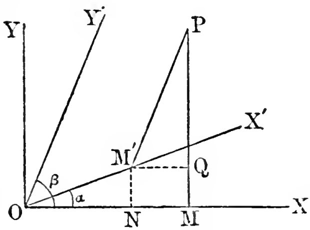
Whence

$$\left.\begin{aligned} x' &= (y - b)\sin\delta + (x - a)\cos\delta; \\ y' &= (y - b)\cos\delta - (x - a)\sin\delta; \end{aligned}\right\} \tag{14}$$

which are the required expressions.

29. *Transformation from rectangular to oblique co-ordinates, the origin remaining the same.*

Let OX, OY be the rectangular axes, and OX', OY' the oblique axes; the angle $XOX' = \alpha$, $XOY' = \beta$; and let x, y be the co-ordinates of any point P referred to the rectangular axes, and x', y' the co-ordinates of the same point referred to the oblique axes. Then



$$\begin{aligned} x = OM &= ON + M'Q \\ &= OM' \cos XOX' + PM' \cos XOY' \\ &\qquad\qquad (\text{since } XOY' = PM'Q) \\ &= x' \cos\alpha + y' \cos\beta, \end{aligned}$$

and

$$\begin{aligned} y = PM &= M'N + PQ \\ &= OM' \sin XOX' + PM' \sin XOY' \\ &= x' \sin\alpha + y' \sin\beta; \end{aligned}$$

which are the expressions of the rectangular co-ordinates in terms of the oblique ones. If we express the oblique co-ordinates in terms of the rectangular ones, we shall have

$$x' = \frac{x \sin \beta - y \cos \beta}{\sin(\beta - \alpha)} \qquad \text{and} \qquad y' = \frac{y \cos \alpha - x \sin \alpha}{\sin(\beta - \alpha)}.$$

Of Loci.

30. The first fundamental principle of Analytic Geometry, as developed in what precedes, may be expressed thus:

Having chosen a system of co-ordinates, then

To every pair of values of the co-ordinates corresponds one definite point in the plane.

Let us now suppose that, instead of the co-ordinates being given, only an equation of condition between them is given. Then we may assign any value we please to one co-ordinate, and find a corresponding value of the other. To every such pair of corresponding values will correspond a definite point. Since these pairs of values may be as numerous as we please, we conclude:

A pair of co-ordinates subjected to a single equation of condition may belong to a series of points unlimited in number.

If one co-ordinate varies continuously and uniformly, the other will vary according to some regular law. From this follows:

The points whose co-ordinates satisfy an equation of condition all lie on one or more lines, straight or curved.

Def. A line, or system of lines, the co-ordinates of every point of which satisfy an equation of condition is called the **locus** of that equation.

31. Problem. *To draw the locus of a given equation.*

Solution. 1. By means of the equation express one co-ordinate, no matter which, in terms of the second.

2. Assign to this second co-ordinate a series of values, at pleasure, differing not much from each other.

3. Find each corresponding value of the other co-ordinate.

4. Lay down the point corresponding to each pair of values thus found, and join all the points by a continuous line.

5. This line will be the required locus.

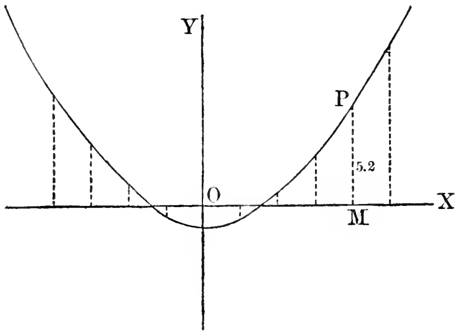
EXAMPLE 1. Construct the locus of the equation

$$10y = x^2 - x - 4.$$

Assigning to x values from -10 to $+10$, differing by two units, we have

$x = -10$	-8	-6	-4	-2	0	$+2$	$+4$	$+6$	$+8$	$+10$
$y = +10.6$	$+6.8$	$+3.8$	$+1.6$	$+.2$	$-.4$	$-.2$	$+.8$	$+2.6$	$+5.2$	$+8.6$

Laying down the positions of these eleven points corresponding to these pairs of co-ordinates, we find them to be as in the annexed diagram.



EXAMPLE 2. Construct the locus of the equation

$$(y - 5)^2 + (x - 12)^2 = 100.$$

From this quadratic equation we obtain for the value of y, in terms of x,

$$y = 5 \pm \sqrt{100 - (x - 12)^2}.$$

The following conclusions follow from this equation:

1. For every value we assign to x there will be two values of y, the one corresponding to the positive, the other to the negative value of the sum. To form the locus we must lay down both of these values.

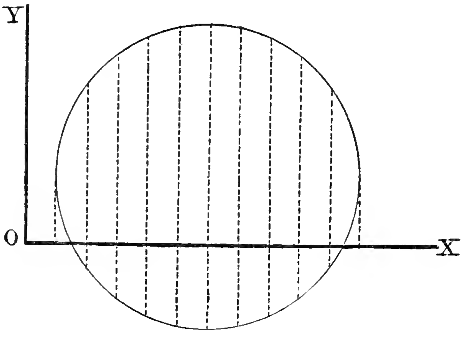
2. If the value of $(x - 12)^2$ exceeds 100, which will be the case when $x < 2$ or $x > 22$, the quantity under the radical sign will be negative, and the value of y will be imaginary. This shows that there is no value of y, and therefore no point of the curve, except when x is contained between the limits

$$2 < x < 22.$$

We now find the following sets of corresponding values of x and y:

$x =$	2	3	4	6	8	10	12	14	16	18	20	21	22
$y =$	5.0	9.4	11.0	13.0	14.2	14.8	15.0	14.8	14.2	13.0	11.0	9.4	5.0
$y =$	5.0	0.6	−1.0	−3.0	−4.2	−4.8	−5.0	−4.8	−4.2	−3.0	−1.0	0.6	5.0

Laying down these points upon a diagram, we shall find them to fall as in the annexed figure.



EXERCISES.

Construct the loci of the following equations to rectangular co-ordinates:

1. $y = 3x^2 - x - 10.$
2. $y = \sin x.$
3. $y = \cos x.$

Note. In the last two exercises we should, in rigor, adopt the unit radius, 57° 18′, as the unit of x. But a more convenient and equally good course will be to take 60° as the unit, and let it correspond to one inch on the paper. Lay off a scale of sixths of an inch on the axis of X, and let the successive points, one sixth of an inch apart, be 0°, 10°, 20°,

$30°, \ldots\ldots 360°$. At each point erect, as an ordinate, the corresponding value of the *natural* sine or cosine, and draw the curve through the extremities. The curve is called the *curve of sines.*

We need not stop at 360°, but may continue on indefinitely. The curve will be a wave-line, the parts of which continually repeat themselves.

4. $y = \frac{x}{3} + 1.$
5. $y = 3x + 1.$
6. $x = \frac{1}{4}y^2 - 3.$
7. $5x = y^2 - 5y - 5.$
8. $10x = y^2 - 7y - 10.$
9. $y^2 = x^2.$
10. $y = \tan x.$
11. $y = \sec x.$

NOTE. The object of the above exercises is to give the student a clear practical idea of the relation between an equation and its locus. He should perform as many of them as are necessary for this purpose. It is in theory indifferent what scale of units of length is used, but in practice a scale either of millimetres or tenths of an inch will be found most convenient.

32. *Intersections of Loci.* Consider the following problem:

To find the point or points of intersection of two loci given by their equations.

Solution. Since the points in question are common to *both* loci, their co-ordinates must satisfy *both* equations. Hence we have to find those values of the co-ordinates which satisfy both equations. This is done by solving the equations algebraically, regarding the co-ordinates as unknown quantities.

If the equations are each of the first degree, there will be but one pair of values of the co-ordinates, and therefore but one point of intersection.

If the equations are one or both of the second or any higher degree, there may be several roots, in which case there will be one point for each pair of roots. The curves will then have several points of intersection.

If the roots are one or both imaginary, the loci will not intersect at all. This is expressed by calling the points of intersection *imaginary.*

Example. To find the point in which the loci whose equations are

$$y^2 + 2x^2 = 164$$

and

$$y = 2x - 3$$

intersect each other.

We have here a pair of simultaneous equations, one of which is a quadratic. Substituting in the first the value of y from the second, we have the quadratic equation in x,

$$6x^2 - 12x = 155.$$

The solution of this equation gives

$$x = 1 \pm \sqrt{\frac{161}{6}} = + 6.18 \quad \text{or} \quad - 4.18.$$

The corresponding values of y are

$$y = 9.36 \quad \text{or} \quad - 11.36.$$

We have in this theory a correspondence between the *mobility* of a point in space and the *variability* of an algebraic quantity, which is at the basis of Analytic Geometry. That is:

To the *unlimited variability* of the co-ordinates x and y corresponds the *mobility* of a point to all parts of a plane.

To the *limited variability* of co-ordinates subjected to one equation of condition corresponds the *limited mobility* of a point confined to a straight or curve line, but at liberty to move anywhere along that line.

To the *constancy* of co-ordinates required to satisfy two equations corresponds the *immobility* of a point required to be on two lines at once, that is, confined to the intersections of two lines.

It must always be understood that liberty to occupy any one of several points, as when the curves have several points of intersection, is not mobility.

EXERCISES.

Find the points of intersection of the following loci:

1. $\frac{x}{a} + \frac{y}{b} = 1$ and $\frac{x}{b} - \frac{y}{a} = 2.$

2. $x^2 + y^2 = 9$ and $\frac{x}{2} + \frac{y}{3} = 1.$

3. $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and $x^2 + y^2 = \frac{5}{4}.$

4. $y^2 = 4ax$ and $x - \sqrt{3}y = 2.$

Do the following loci intersect?—

5. $3x^2 - y^2 = -4$ and $x^2 + y^2 - 2x = 0.$

6. $y - \frac{x}{\sqrt{3}} = \frac{2}{3}\sqrt{29}$ and $9x^2 + 25y^2 = 225.$

NOTE. The special values of the co-ordinates found from the above exercises are constants, the relation of which to the variables may be explained by thinking thus: The co-ordinates are affected by a love of liberty which prompts them to take all possible values so long as we, their masters, do not subject them to any condition.

If we require them to satisfy an equation, they obey us, but exercise their liberty by assuming all values consistent with that equation.

If we require them also to satisfy a second equation, we deprive them of all liberty of variation, and chain them down to the special values which satisfy both equations.

Again, if we put

$$P \equiv ax + by + c,$$

then, so long as we require the co-ordinates to satisfy the equation $P = 0$, P retains this zero value. But if we rub out the $= 0$, and leave only the symbol P without any equation, the co-ordinates instantly resume their liberty, and, by varying, make P take all values whatever.

CHAPTER III.

THE STRAIGHT LINE.

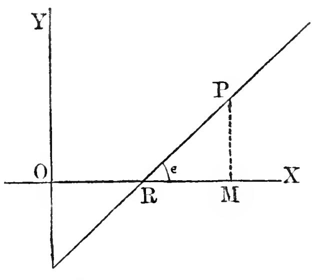
SECTION I. ELEMENTARY THEORY OF THE STRAIGHT LINE.

The Equation of a Straight Line.

33. PROBLEM. *To find the equation of a straight line.*

In order that the equation of any locus may be found, the locus must be so described that the position of each of its points can be determined. Hence, to find the equation of a straight line, we must suppose the data which determine the situation of the line to be given. This may be done in various ways, of which the following are examples:

I. A line is completely determined if the point in which it intersects the axis of X, and the angle which it makes with that axis, are given. Let us then suppose given:



The abscissa $OR \equiv a$ of the point R in which the line intersects the axis of X;

The angle ε which the line makes with that axis.

To find the equation, let P be any point whatever on the line. From P drop the perpendicular PM upon the axis of X. If we then put

x, y, the co-ordinates of P,

we shall have

$$MP = y = RM \tan \varepsilon = (OM - OR) \tan \varepsilon = (x - a) \tan \varepsilon.$$

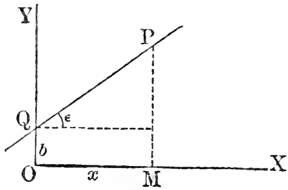
Hence, putting $m \equiv \tan \varepsilon$, we have

$$y = m(x - a). \tag{1}$$

Because P may be any point whatever on the line, this equation must subsist between the co-ordinates of every point of the line; it is therefore the equation of the line.

Def. The **slope** of a line is the tangent of the angle which it forms with the axis of abscissas.

II. Let the slope m of the line, and the ordinate b of the point in which the line cuts the axis of Y, be given.



Using the same notation as before, we readily find

$$MP = x \tan \varepsilon + b,$$

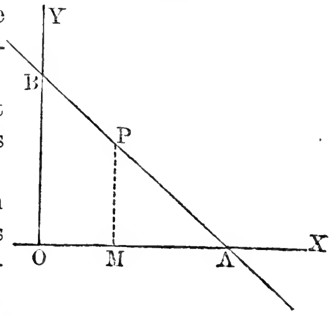
or $$y = mx + b; \tag{2}$$

which last is the required equation.

III. Let the points A and B in which the line intersects the axes of co-ordinates be given. Let us then put

$a \equiv$ the abscissa OA of the point A in which the line intersects the axis of X;

$b \equiv$ the ordinate of the point B in which it intersects the axis of Y.



Then, if P be any point on the line, the similar triangles BOA and PMA give the proportion

$$b : a = PM : MA = y : a - x.$$

We hence derive

$$ay = b(a - x);$$

that is, $$bx + ay = ab,$$

or $$\frac{x}{a} + \frac{y}{b} = 1. \tag{3}$$

Def. The lengths OA and OB from the origin to the points in which the line cuts the co-ordinate axes are called the **intercepts** of the line upon the respective axes.

EXERCISES.

1. Write the equations of lines passing through the origin and making the angles of 45°, 30°, 120°, 135°, 150° and ε, respectively, with the axis of X.

2. If the intercept of a line on the axis of X is a, and if ε is the angle which it makes with that axis, express its intercept on the axis of Y.

3. Write the equation of the line whose intercept on the axis of $Y = 5$ and which makes an angle of 30° with the axis of X.

4. Form the equation of the line whose intercept on the axis of X is a and which makes an angle of 45° with that axis.

5. Show geometrically that the inverse square of the perpendicular from the origin upon a line is equal to the sum of the inverse squares of its intercepts on the axes.

NOTE. The inverse square of a is $1 \div a^2$.

6. Express the tangents of the angles which a line makes with the co-ordinate axes in terms of its intercepts upon those axes, and explain the algebraic sign of the tangent.

7. Two lines have the common intercept a upon the axis of X; the difference of their slopes is unity; and the sum of their intercepts upon the axis of Y is c. Find the separate intercepts upon y, and show that the equations of the two lines are

$$\frac{x}{a} + \frac{2y}{c - a} = 1 \quad \text{and} \quad \frac{x}{a} + \frac{2y}{c + a} = 1.$$

8. What is the relation of the two lines,

$$y = mx + b$$

and

$$y = mx + b + c?$$

9. What are the relations of the series of lines,

$$\begin{aligned} y &= mx, \\ y &= mx + b, \\ y &= mx + 2b, \\ y &= mx + 3b, \\ \text{etc.} &\quad \text{etc.?} \end{aligned}$$

10. What is the relation of the two lines,

$$y = b + mx$$

and

$$y = b - mx?$$

Especially show where they intersect, the relation of the angles they form with the axis, and the angle they form with each other in terms of $\varepsilon = \text{arc tan } m$.

34. The equations (1), (2) and (3) are examples of the numerous forms which the equation of a right line may assume. We have now to generalize these forms.

Def. An **equation of the first degree** between two variables, x and y, means any equation which can be reduced to the form

$$Ax + By + C = 0, \tag{4}$$

A, B and C being any constant quantities whatever.

THEOREM. *Every equation of the first degree between rectangular co-ordinates represents a straight line.*

Proof. The equation (4) may be reduced to the form

$$y = -\frac{A}{B}\left(x + \frac{C}{A}\right). \tag{5}$$

Since the tangent of a varying angle takes all values, we can always find an angle, $\equiv \varepsilon$, whose tangent shall be $-\frac{A}{B}$.

On the axis of X measure a distance $-\frac{C}{A} \equiv a$.

Then, by (1), the locus of (5) will be the line which intersects the axis of X at the point $x = a$ and makes an angle ε with the axis of X. Since such a line is always possible, the theorem is proved.

Scholium. The result of the above theorem may be expressed as follows:

The locus of the equation

$$Ax + By + C = 0$$

is that straight line which intersects the axis of X at the distance $-\frac{C}{A}$ from the origin and makes with that axis an angle whose tangent is $-\frac{A}{B}$.

35. *Reduction of the General Equation.* Any pair of values of x and y which satisfy the equation

$$Ax + By + C = 0$$

must also make

$$m(Ax + By + C) = 0;$$

that is,

$$(mA)\,x + (mB)y + mC = 0.$$

Hence, since m may be any quantity whatever,

If we multiply or divide all the coefficients, A, B and C, which enter into the general equation, by the same factor or divisor, the line represented by the equation will not be altered.

EXAMPLE. The equations

$$\begin{aligned} y - 2x + 1 &= 0, \\ 2y - 4x + 2 &= 0, \\ 5y - 10x + 5 &= 0, \end{aligned}$$

all represent the same line, because they all give the same value of y in terms of x, namely,

$$y = 2x - 1.$$

The same result may be expressed in the form:

The line represented by the equation (4) *depends only on the mutual ratios of the coefficients A, B* and *C, and not upon their absolute values.**

36. From this it follows that special forms of the general equation may be obtained by multiplying or dividing it by any quantity.

I. *First Form.* By dividing by B we obtain

$$\frac{A}{B}x + y + \frac{C}{B} = 0,$$

or

$$y = -\frac{A}{B}x - \frac{C}{B}, \qquad (6)$$

---

* This introduction of more quantities than are really necessary for the expression of a result is quite frequent in Mechanics and Geometry. It has the advantage of enabling us to assign such values to the superfluous quantities as will reduce the expression to the most convenient form.

which becomes identical with the form (2) by putting

$$m \equiv -\frac{A}{B}; \qquad b \equiv -\frac{C}{B}. \tag{7}$$

II. *Second Form.* By dividing by C the general equation becomes

$$\frac{A}{C}x + \frac{B}{C}y + 1 = 0, \tag{8}$$

or

$$-\frac{A}{C}x - \frac{B}{C}y = 1, \tag{9}$$

which becomes identical with (3) by putting

$$a \equiv -\frac{C}{A}; \qquad b \equiv -\frac{C}{B}.$$

III. *Third or Normal Form.* Let us divide by $\sqrt{A^2 + B^2}$. The equation will then become

$$\frac{A}{\sqrt{A^2 + B^2}}x + \frac{B}{\sqrt{A^2 + B^2}}y + \frac{C}{\sqrt{A^2 + B^2}} = 0. \tag{10}$$

If we now determine an angle α by the equation

$$\sin \alpha = \frac{B}{\sqrt{A^2 + B^2}},$$

we shall have

$$\cos \alpha = \sqrt{1 - \sin^2 \alpha} = \frac{A}{\sqrt{A^2 + B^2}}. \tag{11}$$

Let us also put, for brevity,

$$-p = \frac{C}{\sqrt{A^2 + B^2}}.$$

The general equation of the line will then become

$$x \cos \alpha + y \sin \alpha - p = 0, \tag{12}$$

which is called the **Normal form** of the equation of a straight line.

EXERCISES.

Express each of the following equations in the forms (2), (3) and (10):

1. $3x + 4y + 15 = 0.$
2. $4x + 3y - 15 = 0.$
3. $12x - 5y - 13 = 0.$
4. $x - 2y + 6 = 0.$
5. $x + y + c = 0.$
6. $x - y - c = 0.$

37. *Relation of the General Equation to its Special Forms.* The forms (1), (2) and (3) are examples of numerous special forms under which the equation of a straight line may be written. The general form is not to be regarded as a distinct form, but as a form which may be made to express all others by assigning proper values to the constants A, B and C. For example:

The form (1) is equivalent to

$$y - mx + ma = 0,$$

which is what the general form becomes when we put

$$A \equiv -m,$$
$$B = 1,$$
$$C \equiv am.$$

In the same way, to reduce the general form to (2), we have only to put

$$A \equiv -m,$$
$$B = 1,$$
$$C \equiv -b.$$

To reduce it to (3) we put

$$A \equiv \frac{1}{a},$$
$$B \equiv \frac{1}{b},$$
$$C = 1.$$

Again, the normal form is one expressed by the general form when we suppose

$$A \equiv \cos \alpha,$$
$$B \equiv \sin \alpha,$$
$$C \equiv -p.$$

We may also say that the normal form is one in which

$$A^2 + B^2 = 1.$$

38. Since all forms of the equation of a straight line are special cases of the general form, we conclude:

If we demonstrate any theorem by means of the general form of the equation of a straight line, that demonstration will include all the special forms.

39. *Def.* The constants A, B and C which enter into the equation of a straight line are called its **parameters.**

The parameters determine the situation of a line as co-ordinates do the position of a point.

Only two parameters are really necessary to determine the line, but there is often a convenience in using three, as in the general form.

A line is completely determined when its parameters are given. Instead of saying,

"The line whose equation is $Ax + By + C = 0$,"

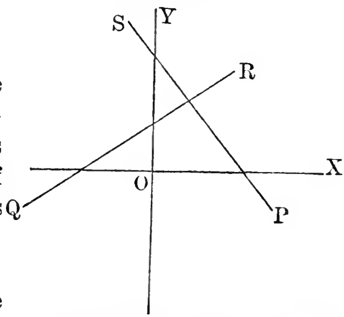
we may say,

"The line (A, B, C)."

40. *Special Cases of Straight Lines.*

I. If, in the general equation of the straight line,

$$Ax + By + C = 0,$$



the coefficients A and B are of opposite signs, x must increase with y, the line makes an acute angle with the axis of X, and its positive direction is in the first or third quadrant.

QR is such a line.

II. If A and B are of the same sign, one co-ordinate diminishes as the other increases, the line makes an obtuse angle with the axis of X, and its positive direction is in the second or fourth quadrant.

PS is such a line.

III. If A vanishes, the equation may be reduced to

$$y = -\frac{C}{B} = \text{a constant},$$

while x may have any value whatever.

The line is then parallel to the axis of X and at the distance $-\frac{C}{B}$ from it.

IV. In the same way, the equation of a line parallel to the axis of Y is

$$x = \text{a constant},$$

the constant being the distance of the line from the axis of Y.

V. If this constant itself vanishes, the line will coincide with the axis of Y. Hence the equation of the axis of y is

$$x = 0.$$

VI. In the same way, the equation of the axis of x is

$$y = 0.$$

EXERCISES.

1. At what point does the line

$$ax + c = 0$$

cut the axis of X?

2. Write the equation of a line perpendicular to the axis of X and cutting off an intercept, b, from that axis.

3. What are the relations of the four lines,

$$x = a; \qquad x = -a;$$
$$y = b; \qquad y = -b;$$

and what figure do they form?

41. *Special Problems connected with the General Equation of a Straight Line.*

I. *To find the intercepts of the general straight line upon the co-ordinate axes.*

By definition, the intercept upon the axis of X is the value of x when $y = 0$. Putting $y = 0$ in the general equation, it becomes

$$Ax + C = 0.$$

Hence, if we put a for the intercept upon the axis of X, we have

$$a = -\frac{C}{A}.$$

In the same way, we find for the intercept on Y, which we call b,

$$b = -\frac{C}{B}.$$

II. *To find the angle which a line makes with the axis of X.*

We have already shown (§ 34) that

$$\tan \varepsilon = -\frac{A}{B}.$$

We can now find the sine and cosine of ε by trigonometric formulæ, as follows:

$$\left.\begin{aligned} \sin \varepsilon &= \frac{\tan \varepsilon}{\sqrt{1 + \tan^2 \varepsilon}} = -\frac{A}{\sqrt{A^2 + B^2}}; \\ \cos \varepsilon &= \frac{1}{\sqrt{1 + \tan^2 \varepsilon}} = \frac{B}{\sqrt{A^2 + B^2}}. \end{aligned}\right\} \quad (13)$$

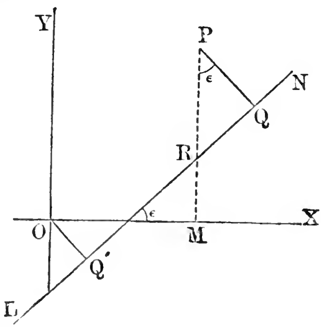
III. *To express the perpendicular distance of a point from a given line.*

Let x' and y' be the co-ordinates of the point, and

$$Ax + By + C = 0$$

the equation of the line.

Since the position of the point is completely determined by its co-ordinates, and the line by its parameters, A, B, C, the required distance admits of being expressed in terms of x', y', A, B and C.



Let P be the point, LN the line, and PQ the perpendicular from the point on the line; and let the ordinate PM of the point intersect the line in R. We shall then have

$$PQ = PR \cos \varepsilon. \quad (a)$$

Now R is a point on the line whose abscissa is the same as that of P, namely, x'; and if we put $RM = y_{,}$ = the ordinate of R, we must have, since R is on the line,

$$Ax' + By_{,} + C = 0,$$

which gives
$$y_{,} = -\frac{Ax' + C}{B}.$$

Then
$$PR = PM - RM = y' - y_{,}$$
$$= \frac{By'}{B} - y_{,} = \frac{Ax' + By' + C}{B}.$$

Substituting in (a) this value of PR and the value of $\cos \varepsilon$ from (13), we have

$$PQ = \frac{Ax' + By' + C}{\sqrt{A^2 + B^2}}. \tag{14}$$

Since the co-ordinates of the origin are $x' = 0$ and $y' = 0$, we have

$$OQ' = \frac{C}{\sqrt{A^2 + B^2}}, \tag{15}$$

which gives the perpendicular from the origin on the line.

EXERCISES.

Find, for each of the lines represented by the following equations,—

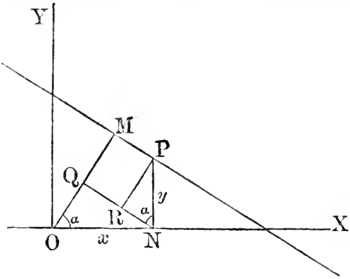
The angle which it makes with the axis of X;
Its intercepts upon the axes;
Its distance from the point (4, 3);
Its least distance from the origin;
The length of that portion intercepted between the axes.

1. $3x + 4y + 10 = 0$.
2. $3x + 4y - 10 = 0$.
3. $5x - 12y + 26 = 0$.
4. $x + y = 0$.
5. $4x - 3y - 5 = 0$.
6. $x - y = 0$.
7. $x \cos \alpha + y \sin \alpha - p = 0$.
8. $\dfrac{x}{a} + \dfrac{y}{b} = 1$.

9. Find the length of the perpendicular from the point (a, b) on the line $\dfrac{x}{a} + \dfrac{y}{b} = 1$, and show that it is equal to the negative distance of the line from the origin.

10. Find the points on the axis of X which are at a perpendicular distance a from the line $\frac{x}{a} + \frac{y}{b} - 1 = 0$.

42. *Direct Derivation of the Normal Form.* This form may be derived as follows: From the origin drop the perpendicular OM upon the line whose equation is required. Let P be any point of the line, and



$x \equiv ON$, the abscissa of P;

$y \equiv NP$, its ordinate;

$\alpha \equiv$ angle NOM of the perpendicular with the axis of X;

$P \equiv OM$.

From N draw NQ parallel to the line, and PR parallel to OM. Then

$$OQ = ON \cos\alpha = x\cos\alpha;$$
$$QM = NP\sin\alpha = y\sin\alpha;$$
$$OQ + QM = p = x\cos\alpha + y\sin\alpha.$$

Hence

$$x\cos\alpha + y\sin\alpha - p = 0,$$

which is the normal form of the equation.

We hence conclude:

In the normal form the parameters p and α are respectively the perpendicular from the origin upon the line, and the angle which this perpendicular makes with the axis of X.

43. *Distances from a Line in the Normal Form.* In this form $A^2 + B^2 = 1$. Hence the distance of the point whose co-ordinates are x' and y' from the line is

$$x'\cos\alpha + y'\sin\alpha - p$$

the same function which, equated to zero, represents the line. Hence the theorem:

If, in the expression $x\cos\alpha + y\sin\alpha - p$, we substitute for x and y the co-ordinates of any point whatever, the expres-

sion will represent the distance of that point from the line whose equation is $x \cos \alpha + y \sin \alpha - p = 0$.

By supposing x' and y' zero, we find the distance of the origin from the line to be $-p$. Since p itself has been taken as essentially positive, we conclude:

The expression for the distance of a point from the line in the normal form is negative when the point is on the same side as the origin, and positive on the opposite side.

This agrees with the convention that the direction *from* the origin *to* the line shall be positive.

EXERCISES.

1. What is the relation of the two lines

$$x \cos 30^\circ + y \sin 30^\circ - p = 0$$

and

$$x \cos 210^\circ + y \sin 310^\circ - p = 0?$$

2. Draw approximately by the eye and hand the lines represented by the following equations:

$$x \cos 30^\circ + y \sin 30^\circ - 5 = 0.$$
$$x \cos 60^\circ + y \sin 36^\circ - 5 = 0.$$
$$x \cos 120^\circ + y \sin 120^\circ - 5 = 0.$$
$$x \cos 240^\circ + y \sin 240^\circ - 5 = 0.$$

Lines Determined by Given Conditions.

When a line is required to fulfil certain conditions, those conditions must be expressed algebraically by equations of condition involving the parameters of the line. The values of the parameters are to be eliminated from the equation of the line by means of these equations of condition.

Since two conditions determine a line, it will be convenient to employ a general form of the equation of the line in which only two parameters appear. Such a form is

$$y = mx + b. \qquad (a)$$

44. *To find the equation of a line which shall pass through a given point and make a given angle with the axis of X.*

Let $(x', y') \equiv$ the given point, and
$\varepsilon \equiv$ the given angle.

One equation of condition is then

$$m = \tan \varepsilon,$$

which determines the parameter m. This gives, for the equation of the line,

$$y = x \tan \varepsilon + b. \tag{b}$$

The condition that the line shall pass through the point (x', y') is

$$y' = mx' + b.$$

To eliminate b, we subtract this equation from (b) after substituting the value of m. This gives

$$y - y' = \tan \varepsilon \, (x - x'),$$

which is the required equation of the line passing through the point (x', y') and making an angle ε with the axis of X. If we write m for $\tan \varepsilon$, it becomes

$$y - y' = m(x - x'), \tag{c}$$

or

$$mx - y - mx' + y' = 0,$$

which, compared with the general form

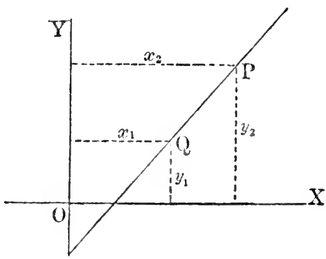
$$Ax + By + C = 0,$$

gives

$$A = m; \qquad B = -1; \qquad C = -mx' + y'.$$

45. *To find the equation of a line passing through two given points.*

Let (x_1, y_1) and (x_2, y_2) be the two given points.



To determine the parameters m and b, we have the conditions

$$\left.\begin{aligned} y_1 &= mx_1 + b, \\ y_2 &= mx_2 + b, \end{aligned}\right\} \tag{d}$$

which give, by subtraction,

$$y_2 - y_1 = (x_2 - x_1)m;$$

whence

$$m = \frac{y_2 - y_1}{x_2 - x_1}. \tag{e}$$

Subtracting the first equation of (d) from (a), we have

$$y - y_1 = m(x - x_1),$$

and substituting the value of m gives

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1), \tag{16}$$

which is the required equation in which the parameters m and b are replaced by the co-ordinates of the given points.

To reduce the equation to the general form, we have, by clearing of denominators,

$$\left.\begin{aligned} A &= y_2 - y_1; \\ B &= x_1 - x_2; \\ C &= y_1(x_2 - x_1) - x_1(y_2 - y_1) = x_2y_1 - x_1y_2. \end{aligned}\right\} \tag{17}$$

Remark. Most of the special forms of the equation already given are cases in which the line is determined by given conditions. For example:

In the form (1) (§ 33) the given quantities are the slope and the intercept on the axis of X.

In the form (2) they are the slope and the intercept on the axis of Y.

In the form (3) they are the two intercepts.

In the Normal form they are the length of the perpendicular from the origin upon the line, and the inclination of the perpendicular to the axis of X.

EXERCISES.

1. Write the equation of a line passing through the point $(-1, 2)$ and making an angle of $135°$ with the axis of X.

2. Write the equation of a line passing through the point $(4, -1)$ and making an angle of $30°$ with the axis of X; find the intercepts which it cuts off from the axes, and the ratios of these intercepts to the length of the line included between the axes.

3. Find the equation of the line passing through the points $(2, 4)$ and $(3, -2)$, and find its intercepts on the axes, the angle which it makes with the axis of X, and its distance from the origin.

4. Find the equation of the line making an angle of $150°$ with the axis of X and passing at a perpendicular distance 5 from the origin.

5. Find the equation of the line passing through the point (1, 5) and intercepting a length 3 on the axis of Y.

6. Find the equation of the line passing through the point (5, − 1) and intercepting a length − 3 on the axis of Y.

7. Write the equations of lines passing through the three following pairs of points:

I. The points (a, b) and $(a, -b)$.
II. The points $(-a, b)$ and (a, b).
III. The points (a, b) and $(-a, -b)$.

8. What is the distance from the point (1, 5) to the line joining the points (− 3, 3) and (1, 6)? *Ans.* $\frac{4}{5}$.

9. If the vertices of a triangle are at the points (1, 3), (3, − 5) and (− 1, − 3), write the equations of the three sides in the general form, and find the distance at which each side passes from the origin.

10. Write the equations of the three medial lines of this last triangle.

NOTE. A medial line of a triangle is the line from either vertex to the middle of the opposite side.

11. Given the co-ordinates of the vertices of a triangle, find the equations of the lines which join the middle points of any two sides, and show that these joining lines are parallel to the sides of the triangle.

12. Find the equations of the three sides of the triangle whose vertices are at the points (a, b), (a', b') and a'', $b'')$. Then find the product of the length of each side into its distance from the opposite vertex, and show that each of these products is equal to the double area of the triangle.

First write the general equation of each side, using the form (17). Then note the relation between each value of $\sqrt{A^2 + B^2}$ and the corresponding side of the triangle. Then form the products and note § 22.

13. Show analytically that if a series of parallel lines are equidistant, they contain between them equal segments of the axes of co-ordinates. Note that the values of p for such lines are in arithmetical progression.

Relation of Two Lines.

46. PROBLEM. *To express the angle between two lines in terms of the parameters of the lines.*

Let the lines be

$$\left.\begin{aligned} Ax + By + C &= 0 \\ \text{and} \quad A'x + B'y + C' &= 0. \end{aligned}\right\} \qquad (a)$$

The angle between them will be the difference of the angles which they make with the axis of X; that is, using the previous notation, it will be $\varepsilon - \varepsilon'$.

The expression for the tangent of $\varepsilon - \varepsilon'$ will be the simplest. We have, by trigonometry,

$$\tan(\varepsilon - \varepsilon') = \frac{\tan\varepsilon - \tan\varepsilon'}{1 + \tan\varepsilon\tan\varepsilon'}. \qquad (18)$$

Substituting the values of $\tan\varepsilon$ and $\tan\varepsilon'$ found from (13), this equation becomes, by reduction,

$$\tan(\varepsilon - \varepsilon') = \frac{A'B - AB'}{AA' + BB'}. \qquad (19)$$

Or, if we put, as before,

$$m \equiv \tan\varepsilon, \qquad m' \equiv \tan\varepsilon',$$

the expression will be

$$\tan(\varepsilon - \varepsilon') = \frac{m - m'}{1 + mm'}. \qquad (20)$$

Either of the forms (18), (19) and (20) is a solution of the problem.

47. The following are special cases of the preceding general problem:

I. *To find the condition that two lines shall be parallel.*

This condition requires that we have

$$\varepsilon - \varepsilon' = 0° \quad \text{or} \quad 180°;$$

that is,

$$\tan(\varepsilon - \varepsilon') = 0.$$

Hence, from (20), the required condition is

$$\left.\begin{aligned} A'B - AB' &= 0, \\ \text{or} \qquad \frac{A'}{B'} &= \frac{A}{B}. \end{aligned}\right\} \qquad (21)$$

II. *To find the condition that two lines shall be perpendicular to each other.*

The lines will be perpendicular when

$$\varepsilon - \varepsilon' = \pm 90^\circ.$$

Then

$$\tan(\varepsilon - \varepsilon') = \infty.$$

In order that the second members of either of the equations (19) or (20) may become infinite, its denominator must be zero. Hence we must have

$$\left.\begin{aligned} AA' + BB' &= 0, \\ \text{or} \qquad 1 + mm' &= 0, \end{aligned}\right\} \qquad (22)$$

or

$$\tan \varepsilon \tan \varepsilon' = -1,$$

which are three equivalent forms.

EXERCISES.

Write the equations of the lines passing through the origin and perpendicular to each of the following lines:

1. $ax + by + c = 0$. *Ans.* $bx - ay = 0$.
2. $y = mx + b$.
3. $a(x + y) - b(x - y) = 0$.
4. $x + ny = c$.
5. $(x - x_0) = m(y - y_0)$.

6. Write the equation of the line passing through the point (a, b) and perpendicular to the line

$$Ax + By + C = 0.$$

7. Write the equation of the line through the point (a, b) parallel to the line

$$Ax + By + C = 0.$$

8. Express the tangent, sine and cosine of the angle between the lines

$$ax + by + c = 0;$$
$$ax - by + c = 0.$$

9. Write the equations of two lines passing through the origin, and each making an angle of 45° with the line

$$ax + by + c = 0.$$

Ans. $(a + b)\,x - (a - b)\,y = 0,$
and $(a - b)\,x + (a + b)\,y = 0.$

10. Compute the interior angles of the triangle the equations of whose sides are

$$\begin{aligned} x - 2y + 7 &= 0; \\ x + \ \ y - 3 &= 0; \\ x + 3y \qquad &= 0. \end{aligned}$$

11. If the co-ordinates of the three vertices of a triangle are (2, 5), (2, − 3), (4, − 1), it is required to find the equations of the three perpendiculars from the vertices upon the opposite sides.

12. Find the equations of the perpendicular bisectors of the sides of the same triangle.

13. Show that the lines joining the middle points of the consecutive sides of a quadrilateral form a parallelogram.

To do this assume symbols for the co-ordinates of the four vertices; then express the middle points of the sides by § 23, and then the equations of the joining lines by § 45, and show that opposite lines are parallel.

14. Find the condition that the lines

$$x \cos \alpha + y \sin \alpha - p = 0 \quad \text{and} \quad x \sin \beta - y \cos \beta - p'$$

may be parallel.

15. If two lines intersect each other at right angles, and if a and b be the intercepts of the one line, and a' and b' of the other, it is required to show:

(α) That of the four quantities, a, b, a', b', either three will be positive and one negative, or three negative and one positive.

(β) That these quantities satisfy the condition

$$aa' + bb' = 0.$$

16. What is the rectangular equation of the line whose polar equation is

$$\frac{6}{r} = 4 \cos \theta + 3 \sin \theta ?$$

17. Find the area of the triangle formed by the straight lines

$$y = x \tan 75^\circ, \qquad y = x, \qquad y = x \tan 30^\circ + 2.$$

18. Reduce $3r \cos \theta - 2r \sin \theta = 7$ to the form

$$r \cos (\theta - \alpha) = p,$$

and find the values of α and p.

19. Show that if $b^2 - a^2 = 1$, the lines

$$x + (a + b)y + c = 0 \quad \text{and} \quad (a + b)x + (a^2 - b^2)y + d = 0$$

are perpendicular to each other.

20. Show that when the axes are oblique, the ratio $x : y$ of the two co-ordinates of a point is equal to the ratio

Dist. from axis of Y : Dist. from axis of X.

21. Show that the lines $x + y = a$ and $x - y = a$ are at right angles, whatever be the axes.

22. Show that the locus of a point equidistant from two straight lines is the bisector of the angle they form.

48. *To find the point of intersection of two lines given by their equations.*

As already shown, the co-ordinates of the point of intersection are those values of x and y which satisfy *both* equations, (§ 32). If the given equations are

$$Ax + By + C = 0,$$
$$A'x + B'y + C' = 0,$$

we find, for the values of the co-ordinates,

$$x = \frac{BC' - B'C}{AB' - A'B};$$

$$y = \frac{A'C - AC'}{AB' - A'B};$$

which are the required co-ordinates of the point of intersection.

Remark. The preceding result affords another way of deducing the condition of parallelism by the condition that two lines are parallel when their point of intersection recedes to infinity. Let the student find this as an exercise.

49. *To find the condition that three straight lines shall intersect in a point.*

The required condition must be expressed in the form of an equation of condition between the nine parameters of the three lines. Let the equations of the lines be

$$\begin{aligned} ax + by + c &= 0; \\ a'x + b'y + c' &= 0; \\ a''x + b''y + c'' &= 0. \end{aligned}$$

If the three lines intersect in a point, there must be *one* pair of values of x and y which satisfy all three equations. By the last section we have, for the co-ordinate y of the point of intersection of the first two lines,

$$y = \frac{a'c - ac'}{ab' - a'b},$$

and of the last two,

$$y = \frac{a''c' - a'c''}{a'b'' - a''b'}.$$

If the three lines intersect in a point, these values of y must be equal. Equating them and reducing, we find

$$c(a'b'' - a''b') + c'(a''b - ab'') + c''(ab' - a'b) = 0,$$

which is the required equation of condition.

EXERCISES.

1. Given the three lines

$$\begin{aligned} x + 2y + 4 &= 0, \\ 2x - y - 7 &= 0, \\ 3x + y + c &= 0, \end{aligned}$$

it is required to determine the constant c so that the lines shall intersect in a point, and to find the point of intersection.

Ans. $c = -3$.
Point $= (2, -3)$.

2. Express the condition that the three lines

$$\begin{aligned} y &= mx + c, \\ y &= m'x + c', \\ y &= m''x + c'', \end{aligned}$$

shall intersect in a point.

3. Find the point of intersection of the two lines

$$y = mx + c,$$
$$y = m'x - c.$$

4. Prove algebraically that if two lines are each parallel to a third, they are parallel to each other.

NOTE. We do this by showing that from the equations

$$ab' - a'b = 0,$$
$$ab'' - a''b = 0,$$

follows

$$a'b'' - a''b' = 0.$$

5. If the equations of the four sides of a parallelogram are

$$y = mx + c,$$
$$y = m'x - c,$$
$$y = mx + c',$$
$$y = m'x - c',$$

it is required to find the co-ordinates of its four vertices and the equations of its diagonals.

Ans., in part. Equations of diagonals:

$$y = \frac{m + m'}{2} x;$$

$$x = \frac{c + c'}{m' - m}.$$

6. What relation must exist among a, a', m and m' that the lines

$$y = mx + a,$$
$$y = m'x - a',$$

may intersect on the axis of X? *Ans.* $a'm + am' = 0.$

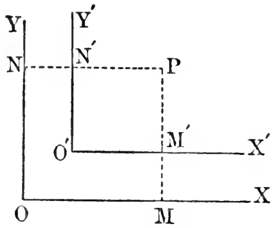
50. *Transformation to New Axes of Co-ordinates.*

By the formulæ of § 25, the equation of a line referred to one system of co-ordinates may be changed to another system by an algebraic substitution.

To make the change it is necessary to express the co-ordinates of the original system in terms of those of the new system, and to substitute the expressions thus found in the equation of the locus.

EXAMPLE I. Let

$$Ax + By + C = 0 \qquad (a)$$



be the equation of a line referred to the system (X, Y).

Let it be required to refer the line to a system (X', Y') parallel to the first and having the origin O' at the point (a, b).

By § 26, the expressions for the original co-ordinates in terms of the new ones will be

$$x = x' + a;$$
$$y = y' + b.$$

Substituting these values in the equation (a), we find for the equation of the line, in terms of the new co-ordinates,

$$Ax' + By' + Aa + Bb + C = 0.$$

The coefficients A and B of the co-ordinates remain unchanged, showing that the line makes the same angle with the new axes as with the old ones.

EXAMPLE II. Let the new system of co-ordinates have the same origin, but a different direction.

The equations of transformation are then (3) of § 27. Substituting the values of x and y there given in the equation (a) of the preceding example, we have

$$(A \cos \delta + B \sin \delta)x' + (B \cos \delta - A \sin \delta)y' + C = 0.$$

The sum of the squares of the coefficients of x' and y' reduces to $A^2 + B^2$, as it should.

EXERCISES.

1. What will be the equation of the line

$$y = 2x + 5$$

when referred to new axes, parallel to the original ones, having their origin at the point (2, 3)?

2. What change must be made in the direction of the axis of X that the line whose equation is $x = y$ may be represented by the equation $x' = 2y'$?

SECTION II. USE OF THE ABBREVIATED NOTATION.*

51. *Functions of the Co-ordinates.* We call to mind that, corresponding to any point we choose to take in the plane, there will be a definite value of each of the co-ordinates x and y. Hence if we take any function of x and y, such, for example, as

$$P \equiv x + 2y + 1,$$

this function P will have a definite value for each point of the plane, which value is formed by substituting in P the values of the co-ordinates for that point. We may then imagine that on each point is written the value of P corresponding to that point.

EXAMPLE.

3	4	5	6	7	8
1	2	3	4	5	6
−1	0	1	2	3	4
−3	−2	−1	0	1	2
−5	−4	−3	−2	−1	0

The above scheme shows the values of the preceding function $P \equiv x + 2y + 1$ for a few equidistant points, assuming the common distance between the consecutive numbers on each line to be the unit of length.

52. *Isorropic Lines.* We may imagine lines drawn through all points for which P has the same value, and may call these lines *isorropic;* that is, lines of equal value. We now have the theorem:

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* This section can be omitted without the student being thereby prevented from going on with subsequent chapters. But, owing to the elegance of the abbreviated notation, the subject, which is not at all abstruse, is recommended to all having mathematical taste.

If the function P is of the first degree in x and y, the isorropic lines will form a system of parallel straight lines.

Proof. Let

$$P \equiv ax + by + c;$$

and let us inquire for what points P has the constant value k. These points will be those whose co-ordinates satisfy the condition

$$P - k = 0,$$

or

$$ax + by + c - k = 0. \tag{a}$$

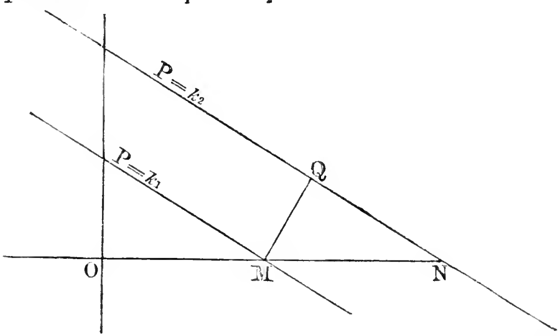
This equation, being of the first degree, is the equation of a straight line, whose angle with the axis of X is given by the equation

$$\tan \varepsilon = -\frac{a}{b}.$$

Since a and b retain the same values, whatever values we assign to k, ε has the same value for each line of the system, and all the lines are parallel.

53. *Distance between Two Lines of the System.* To each value of k in the equation (a) will correspond a certain line. We now have the problem:

To find the distance between the two lines for which P has the respective values k_1 and k_2.



Let OM and ON be the respective intercepts of the lines on the axis of X. We shall then have

$$\text{Distance } MQ = MN \sin \varepsilon.$$

Putting $y = 0$ in the two equations

$$\left.\begin{aligned} ay + bx + c - k_1 &= 0 \\ \text{and} \qquad ay + bx + c - k_2 &= 0, \end{aligned}\right\} \tag{25}$$

we have, for the intercepts,

$$OM = \frac{k_1 - c}{b};$$

$$ON = \frac{k_2 - c}{b};$$

$$\therefore \qquad MN = ON - OM = \frac{k_2 - k_1}{b},$$

and $$MQ = (k_2 - k_1)\frac{\sin \varepsilon}{b} = \frac{k_2 - k_1}{\sqrt{a^2 + b^2}}. \qquad (\S\ 36)$$

Hence, *the distance apart of two isorropic lines is proportional to the difference between the values of P.*

54. *Distance of a Point from a Line.* Let us now return to the general expression

$$P \equiv ax + by + c, \tag{a}$$

and let us study its relation to the line

$$\left.\begin{aligned} ax + by + c &= 0; \\ \text{that is, to the line} \qquad P &= 0. \end{aligned}\right\} \tag{b}$$

In (a) we may suppose x and y to have any values whatever. But in (b) x and y are restricted to those values which correspond to the different points of the line (a, b, c).

Now from what has just been shown it follows that the points for which, in (a), P has the special value k all lie on a straight line parallel to the line $P = 0$, and distant from it by the quantity

$$\frac{k}{\sqrt{a^2 + b^2}}.$$

Hence, if x_0 and y_0 be the co-ordinates of any point at pleasure, we have

$$\text{Distance of point } (x_0, y_0) \text{ from line } (a, b, c) = \frac{ax_0 + by_0 + c}{\sqrt{a^2 + b^2}},$$

a result already obtained in § 41.

These results may be summed up in a third fundamental principle of Analytic Geometry, as follows:

If

$$P \equiv ax + by + c$$

be any function of the co-ordinates of the first degree, then—

I. *To every point on the plane will correspond one definite value of P.*

II. *This value of P is equal to the perpendicular distance of the point from the line $P = 0$ multiplied by the constant factor $\sqrt{a^2 + b^2}$.*

If the expression P is in the normal form, we have

$$a^2 + b^2 = 1, \qquad (\S\ 36)$$

and the factor last mentioned becomes unity.

Hence—

III. *If we have a function of x and y of the form*

$$x \cos \alpha + y \sin \alpha - p \equiv P,$$

this function will express the perpendicular distance of the point whose co-ordinates are x and y from the line $P = 0$.

EXERCISES.

1. Let the student draw the line

$$2x - 3y + 1 = 0,$$

and let him compute the values of the expression

$$2x - 3y + 1$$

for a number of points, and lay them down, as in the scheme of § 51, until he sees clearly the truth of all the preceding conclusions.

2. Imagine a plane covered with values of the function

$$P \equiv ax + by + c,$$

as in § 51. Around the origin as a centre we describe a circle of arbitrary radius, and on its circumference mark the points where the values of P which it meets are greatest and least. Show that all points thus marked lie on the line $bx - ay = 0$.

Show also on what line the points will fall if the centre of the circle is at the point (p, q).

Theorems of the Intersection of Lines.

55. We represent by the symbols P, P', etc., Q, Q', etc., different linear functions of the co-ordinates; e.g.,

$$\begin{aligned} P &\equiv ax + by + c; \\ P' &\equiv a'x + b'y + c'; \\ P'' &\equiv a''x + b''y + c''; \\ &\text{etc.} \qquad \text{etc.} \qquad \text{etc.} \end{aligned}$$

Also, we shall represent by the symbols M, M', etc., N, N', etc., such functions reduced to the normal form in which

$$a^2 + b^2 = 1.$$

Since the M's, N's, etc., will be a special case of the P's, Q's, etc., every theorem true of all the latter will also be true of the former; but the reverse will not always be the case.

The line corresponding to the equation

$$P = 0$$

may, for brevity, be called *the line P.*

56. THEOREM. *If*

$$P = 0, \qquad P' = 0$$

be the equations of any two straight lines, and if μ and ν be any two factors which do not contain x or y, then the equation

$$\mu P + \nu P' = 0 \tag{b}$$

will be that of a third straight line passing through the point of intersection of the lines P and P'.

Proof. 1. By substituting in (b) for P and P' the expressions which they represent, we see that $\mu P + \nu Q$ is a function of the first degree in x and y.

Hence (b) is the equation of *some* straight line.

2. That point whose co-ordinates satisfy both of the equations $P = 0$ and $P' = 0$ must also give $\mu P + \nu P' = 0$, and must therefore lie on the line (b).

But such point is the point of intersection of the lines P and P'.

Hence the point of intersection lies on the line (b), and (b) passes through that point. Q. E. D.

Corollary. If three functions, P, P' and P'', are so related that we can find three factors, λ, μ and ν, which satisfy the identity

$$\lambda P + \mu P' + \nu P'' \equiv 0,$$

then the three lines $P = 0$, $P' = 0$ and $P'' = 0$ *intersect in a point.*

For we derive from this identity

$$P \equiv -\left(\frac{\mu}{\lambda} P' + \frac{\nu}{\lambda} P''\right);$$

whence, by the theorem, P passes through the point of intersection of P' and P''.

57. Theorem. Conversely,

If $P = 0$, $P' = 0$ and $P'' = 0$ are the equations of three lines intersecting in a point, it always will be possible to find three coefficients, μ, ν and λ, *such that*

$$\mu P + \nu P' + \lambda P'' \equiv 0.$$

Proof. 1. Let the values of the three functions P, P' and P'' be

$$\left.\begin{aligned} P &\equiv ax + by + c = 0; \\ P' &\equiv a'x + b'y + c' = 0; \\ P'' &\equiv a''x + b''y + c'' = 0. \end{aligned}\right\} \qquad (a)$$

2. Let us now suppose

$$\left.\begin{aligned} \lambda &\equiv a'b'' - a''b'; \\ \mu &\equiv a''b - ab''; \\ \nu &\equiv ab' - a'b; \end{aligned}\right\} \qquad (b)$$

and let us form the expression $\lambda P + \mu P' + \nu P''$. In this expression we shall have

$$\begin{aligned} \text{Coefficient of } x &= a(a'b'' - a''b') + a'(a''b - ab'') \\ &\qquad + a''(ab' - a'b)\ (\equiv 0); \\ \text{Coefficient of } y &= b(a'b'' - a''b') + b'(a''b - ab'') \\ &\qquad + b''(ab' - a'b)\ (\equiv 0); \\ \text{Absolute term} &= c(a'b'' - a''b') + c'(a''b - ab'') \\ &\qquad + c''(ab' - a'b). \end{aligned}$$

Because the three lines pass through a point, this absolute term is zero (§ 49). Hence the whole expression is identically zero, and the values (b) of λ, μ and ν satisfy the conditions of the theorem.

EXERCISES.

1. Show that if the equations $P = 0$ and $Q = 0$ are so related that we can find two coefficients, μ and ν, which form the identity

$$\mu P + \nu Q \equiv 0,$$

then the two lines P and Q are coincident. (Comp. §§52, 53.)

2. Having the two lines

$$y - mx + a = 0,$$
$$y + mx + 2a = 0,$$

it is required to find the equation of a third line passing through their point of intersection and through the origin.

Method of Solution. Calling the given expressions equated to zero P and Q, and noting that the equation of every line through the point of intersection may be expressed in the form

$$\mu P + \nu Q = 0,$$

we are to determine the quantities μ and ν so that this line shall pass through the origin. Hence the absolute term must vanish. This gives the condition

$$\mu = -2\nu,$$

the value of ν being arbitrary. Substituting this value of μ, and dividing by ν, we find the required equation,

$$y - 3mx = 0.$$

3. Find the equation of a line passing through the origin and through the point of intersection of the lines

$$y - 2x - a = 0;$$
$$y + 2x + 3a = 0.$$

4. Find the equations of the lines making angles of 45° and 135° respectively with the axis of X and passing through the point of intersection of the above two lines.

58. To complete and apply the preceding theory, it is necessary to distinguish between the positive and negative sides of a line. If distances measured on one side are positive, those on the other side are negative. But no rule is possible

for the positive and negative sides without some convention, because the function P may change its sign without changing the position of the line. For example, the two equations

$$x - ny + h = 0,$$
$$-x + ny - h = 0,$$

represent the same line; but all values of x and y which make the one function equal to $+P$ will make the other equal to $-P$, so that the positive and negative sides of the lines are interchanged by the change of form.

Now, in the first form, the distance of the origin from the line is

$$\frac{h}{\sqrt{1 + n^2}}.$$

Hence,

When the absolute term in the equation is positive, the positive side of the line is that on which the origin is situated, and vice versa.

In the normal form the absolute term is negative. Hence,

In the normal form a positive value of the function

$$M \equiv x \cos \alpha + y \sin \alpha - p$$

indicates that the point whose co-ordinates are x and y is on the opposite side of the line from the origin, and a negative value that it is on the same side as the origin.

59. THEOREM. *If*

$$M = 0, \qquad N = 0$$

are the equations of any two lines in the normal form, then the equations

$$M + N = 0, \qquad M - N = 0,$$

will be the equations of the bisectors of the four angles which the lines M and N form at their point of intersection.

Proof. 1. Because the functions M and N are in the normal form, they represent the respective distances of any point from the lines $M = 0$ and $N = 0$. (§ 54.)

2. Every pair of co-ordinates which fulfil the condition

$$M \pm N = 0$$

must give

$$M = \pm N,$$

so that the point which they represent is equally distant from the lines M and N.

3. By geometry, the locus of the point equally distant from two lines is the bisectors of the angles formed by the lines.

REMARK 1. This theorem holds equally true of the equations of any two lines in which the sums of the squares of the coefficients of x and y are equal. For if, in the equations

$$P \equiv ax + by + c = 0,$$
$$P' \equiv a'x + b'y + c' = 0,$$

we have $a^2 + b^2 = a'^2 + b'^2$, then, by § 53, the functions P and P', when not restricted to zero, express the distances of a point (x, y) from the respective lines P and P', multiplied by $\sqrt{a^2 + b^2}$ and $\sqrt{a'^2 + b'^2}$ respectively.

Now, when these multipliers are equal, every point whose co-ordinates satisfy the equation

$$(a \pm a')\, x + (b \pm b')\, y + c \pm c' = 0$$

or

$$P \pm P' = 0$$

must be equally distant from the lines P and P'.

REMARK 2. The equation

$$M - N = 0$$

will be that of the bisector of the angle in which the origin is situated, and of its opposite angle; while the equation

$$M + N = 0$$

will represent the bisector of the two adjacent angles.

EXERCISES.

Find the bisectors of the angles formed by the following pairs of lines:

1. $x - 2y = 0$ and $2x - y = 0$.
2. $y + nx - c = 0$ and $ny - x + c = 0$.
3. Prove the theorem of geometry that the two bisectors of the angles formed by a pair of intersecting lines are at right angles to each other.

In other words, if the functions P and P' are such that

$$a^2 + b^2 = a'^2 + b'^2,$$

then show that the two lines

$$P + P' = 0 \quad \text{and} \quad P - P' = 0$$

intersect at right angles.

4. Show that if $N = 0$ and $N' = 0$ are the equations of two lines in the normal form, then

$$\left.\begin{aligned} \lambda N + \mu N' = 0, \\ \lambda N - \mu N' = 0, \end{aligned}\right\} \qquad (a)$$

will represent the loci of those points whose distances from N and N' are in the ratio $\mu : \lambda$. Also, show geometrically that such a locus is a straight line.

5. In the preceding exercise, what condition must the coefficients λ and μ satisfy in order that the equations (a) may each be in the normal form?

60. *Applications of the Preceding Theorems.* The preceding theorems enable us to prove with great elegance the leading theorems of the intersections of certain lines in a triangle.

I. *The bisectors of the interior angles of a triangle meet in a point.*

Proof. Let

$$L = 0, \qquad M = 0, \qquad N = 0,$$

be the equations of the sides of the triangle.

We suppose the origin to be within the triangle, because we can always move it thither by a transformation of co-ordinates.

Then, by what precedes,

$$\begin{aligned} P &\equiv L - M = 0, \\ P' &\equiv M - N = 0, \\ P'' &\equiv N - L = 0, \end{aligned}$$

will be the equations of the bisectors. But these functions, P, P' and P'', fulfil the identity

$$P + P' + P'' \equiv 0,$$

and reduce to the form § 56 when we suppose

$$\lambda = \mu = \nu = 1.$$

Hence P, P' and P'' all pass through a point.

II. *The bisectors of any two exterior angles and of the third interior angle meet in a point.*

Proof. The equations of two exterior bisectors and of the third interior bisector are

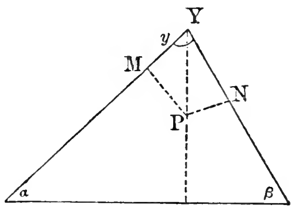
$$\begin{aligned} P &\equiv L + M = 0; \\ P' &\equiv M + N = 0; \\ P'' &\equiv L - N = 0; \end{aligned}$$

which fulfil the identity

$$P - P' - P'' \equiv 0.$$

III. *The perpendiculars from the three vertices of a triangle upon the opposite sides meet in a point.*

Let P be any point upon the perpendicular from Y upon $\alpha\beta$; $PM \perp Y\alpha$, $PN \perp Y\beta$; and $\gamma \equiv$ angle $\alpha Y\beta$.



Then, because the angles $PY\alpha$ and α are complementary,

$$\begin{aligned} PM &= PY \cos \alpha, \\ PN &= PY \cos \beta; \end{aligned}$$

$$\therefore \quad PM : PN = \cos \alpha : \cos \beta.$$

Therefore, if the equations of the sides $Y\alpha$ and $Y\beta$ are

$$N = 0 : N' = 0,$$

then, by the theorem of § 59, Ex. 4, the equation of the perpendicular YP will be

$$N \cos \beta - N' \cos \alpha = 0.$$

In the same way, if the equation of $\alpha\beta$ is $N'' = 0$, we shall have, for the equations of the other two perpendiculars,

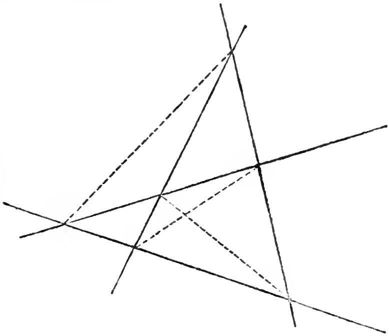
$$\begin{aligned} N' \cos \alpha - N'' \cos \gamma &= 0; \\ N'' \cos \gamma - N \cos \beta &= 0. \end{aligned}$$

The sum of these three equations is identically zero, thus showing that the three lines intersect in a point.

61. *Diagonals of a Quadrilateral.* An elegant and instructive application of the preceding theory is given by the following problem:

To find the equations of the diagonals of a quadrilateral of which the equations of the four sides are given.

We remark that, in general geometry, a quadrilateral has three diagonals. The reason is that each side is supposed to be of indefinite length, and so to intersect the three others. A diagonal is then defined as the line joining the point of intersection of any two sides to the point of intersection of the other two sides. The number of points of intersection, or vertices, is equal to the combinations of two in four, or 6. Taken in pairs these 6 points have three junction lines, as shown in the figure.



Solution. Let the equations of the four sides be

$$\left.\begin{aligned} P &= 0;\\ Q &= 0;\\ R &= 0;\\ S &= 0. \end{aligned}\right\} \qquad (a)$$

We seek for four factors, $\varkappa$, λ, μ and ν, by which to form the identity

$$\varkappa P + \lambda Q + \mu R + \nu S \equiv 0. \qquad (b)$$

Four such factors can always be found when the parameters of P, Q, etc., are given, because by equating to zero the coefficients of x and y and also the absolute term in (b) we shall have three equations which determine any three of the four factors $\varkappa$, λ, μ and ν in terms of the fourth. To the latter we may assign any value at pleasure.

The identity (b) being satisfied, we shall have

$$\varkappa P + \lambda Q \equiv -(\mu R + \nu S). \qquad (c)$$

Now, (§ 56),

$$\varkappa P + \lambda Q = 0$$

is the equation of *some* line passing through the intersection of P and Q, while

$$\mu R + \nu S = 0$$

is the equation of *some* line passing through the intersection of R and S.

But, by (*c*), these two lines are identical. Hence this common line is a diagonal of the quadrilateral. We show in the same way that

$$\varkappa P + \mu R = 0 \qquad \text{or} \qquad \lambda Q + \nu S = 0$$

is the equation of the diagonal joining the intersection of P and R to that of Q and S. Also, that

$$\varkappa P + \nu S = 0 \qquad \text{or} \qquad \lambda Q + \mu R = 0$$

is the equation of the diagonal joining the intersection of P and S to that of Q and R.

Example. To find the equations of the diagonals of the quadrilateral whose sides are

$$\begin{aligned} P &\equiv x + y + 1 = 0; \\ Q &\equiv x + 2y - 3 = 0; \\ R &\equiv x - 2y + 4 = 0; \\ S &\equiv 2x - y - 2 = 0. \end{aligned}$$

Forming the expression (*b*), we find it to be

$$(\varkappa+\lambda+\mu+2\nu)x + (\varkappa+2\lambda-2\mu-\nu)y + \varkappa-3\lambda+4\mu-2\nu \equiv 0.$$

Hence, to form this identity, (§ 8),

$$\begin{aligned} (1)\quad & \varkappa + \lambda + \mu + 2\nu = 0; \\ (2)\quad & \varkappa + 2\lambda - 2\mu - \nu = 0; \\ (3)\quad & \varkappa - 3\lambda + 4\mu - 2\nu = 0. \end{aligned}$$

We solve as follows:

$$\begin{aligned} (2)-(1)\quad & \lambda - 3\mu - 3\nu = 0; \\ (2)-(3)\quad & 5\lambda - 6\mu + \nu = 0. \\ \hline & 3\lambda + 7\nu = 0; \\ & 9\mu + 16\nu = 0. \\ \hline & \lambda = -\frac{7}{3}\nu = -\frac{21}{9}\nu; \\ & \mu = -\frac{16}{9}\nu; \\ & \varkappa = \frac{19}{9}\nu. \end{aligned}$$

The value of ν is arbitrary, and values of κ, μ and λ, free from fractions, are obtained by putting $\nu = 9$. The values of the four coefficients are then

$$\kappa = 19; \qquad \lambda = -21; \qquad \mu = -16; \qquad \nu = 9.$$

From these coefficients the equations of the diagonals are formed by the preceding formulæ, and are found to be

$$\begin{aligned} 2x + 23y - 82 &= 0; \\ x + 17y - 15 &= 0; \\ 37x + 10y + 1 &= 0. \end{aligned}$$

62. *Fundamental Lines of a Triangle.* Let us consider the following problem:

If the equations of the three sides of a triangle in the normal form are

$$\begin{aligned} M &= 0, \\ M' &= 0, \\ M'' &= 0, \end{aligned}$$

what line is represented by the equation

$$M + M' + M'' = 0? \qquad (a)$$

Solution. If we put

$$Q = M + M',$$

the equation $Q = 0$ will represent the bisector of the exterior angle between the lines M and M' (§ 59).

Also, the equation

$$Q + M'' = 0,$$

which is the same as (a), will represent *some* line passing through the point of intersection of M'' and Q, that is, through the point in which the bisector meets the opposite side.

In the same way it may be shown that the line (a) passes through each of the other two points in which the bisectors of the exterior angles meet the opposite sides.

Hence the solution of the problem leads to the theorem:

The three points in which the bisectors of the exterior

angles of a triangle meet the opposite sides lie in a straight line, namely, the line whose equation is

$$M + M' + M'' = 0;$$

$M = 0$, $M' = 0$ *and* $M'' = 0$ *being the equations of the sides in the normal form.*

We may show in the same way that the three equations

$$\begin{aligned} M + M' - M'' &= 0, \\ M - M' + M'' &= 0, \\ -M + M' + M'' &= 0, \end{aligned}$$

are the equations of three straight lines each containing the foot of one bisector of an exterior angle and two bisectors of the two remaining interior angles.

EXERCISES.

1. Show by the preceding theorems that if we form a triangle by joining the points in which each bisector of an interior angle meets the opposite side, the sides of this triangle will severally pass through the points in which the bisectors of the exterior angles meet the opposite sides.

2. Show that if

$$M = 0, \qquad M' = 0, \qquad M'' = 0, \qquad M''' = 0,$$

be the equations of the four sides of a quadrilateral in the normal form, then

$$M + M' + M'' + M''' = 0$$

will be the equation of a straight line containing the three points in which the external bisectors of the three pairs of opposite vertices meet each other.

3. Find the equations of the three diagonals of the quadrilateral whose sides are

$$\begin{aligned} y &= x; \\ y &= x + b; \\ x &= a; \\ y &= -x. \end{aligned}$$

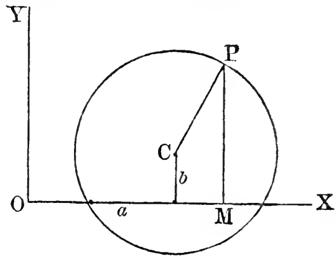
CHAPTER IV.

THE CIRCLE.

Section I. Elementary Theory.

Equation of a Circle.

63. Problem. *To find the equation of a circle.**



Let the co-ordinates of the centre C of the circle be a and b, and let P be any point of the circle.

Calling x and y the co-ordinates of P, we have, for the square of the distance between C and P,

$$CP^2 = (x - a)^2 + (y - b)^2. \qquad (\S\ 17)$$

The condition that P shall lie on the circle requires that this distance shall be equal to the radius of the circle. Let us put

$$r \equiv CP, \text{ the radius of the circle.}$$

The condition then becomes

$$(x - a)^2 + (y - b)^2 = r^2, \qquad (1)$$

which is the required equation of the circle.

64. Theorem. *Every equation between rectangular co-ordinates of the form*

$$m(x^2 + y^2) + px + qy + h = 0 \qquad (2)$$

---

* In the almost universal notation of the higher geometry the word "circle" is used to designate the closed curve which, in elementary geometry, is called the *circumference* of the circle.

in which the coefficients of x^2 and y^2 are equal, while there is no term in xy, represents a circle.

Proof. Divide by m, and put, for brevity,

$$a \equiv -\frac{p}{2m},$$

$$b \equiv -\frac{q}{2m},$$

and the equation will be transformed into

$$x^2 - 2ax + y^2 - 2by + \frac{h}{m} = 0,$$

or

$$(x - a)^2 + (y - b)^2 - a^2 - b^2 + \frac{h}{m} = 0,$$

or

$$(x - a)^2 + (y - b)^2 = a^2 + b^2 - \frac{h}{m}. \tag{3}$$

The first member represents the square of the distance between the fixed point (a, b) and the varying point (x, y). The second member being a constant, the equation shows that the square of the distance of the two points is a constant, whence the distance itself is a constant. Hence the equation represents a circle whose centre is at the point (a, b) and whose radius is $\sqrt{a^2 + b^2 - \frac{h}{m}}$.

65. *Special Forms of the Equation of a Circle.*

We may suppose a circle moved so that its centre shall occupy any required position without the form or magnitude of the circle being changed.

If the centre be at the origin, we have $a = 0$ and $b = 0$, and the equation of the circle becomes

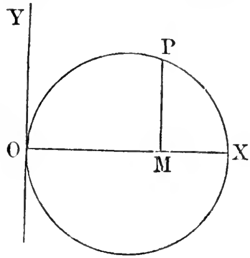
$$x^2 + y^2 = r^2. \tag{4}$$

If the centre is on the axis of X, we have $b = 0$, and the equation becomes

$$y^2 + (x - a)^2 = r^2,$$

which is the equation of a circle whose centre is on the axis of X.

If we suppose $a = r$ and $b = 0$, the circle will be tangent to the axis of Y at the origin, and the equation will become



$$y^2 = r^2 - (x^2 - 2ax + r^2)$$
$$= 2ax - x^2, \qquad (5)$$

which we may define as the equation of a circle when a diameter is taken as the axis of X and the origin is at the end of this diameter.

EXERCISES.

Find the radii and the co-ordinates of the centres of circles having the following equations:

1. $x^2 + y^2 - 10x + 2y + 17 = 0.$
2. $3x^2 + 3y^2 + 6x - 12y - 9 = 0.$
3. $2x^2 + 2y^2 + 8x - 18y - \frac{3}{2} = 0.$
4. $mx^2 + my^2 + px + qy - \frac{pq}{2m} = 0.$

5. Write the equation of the circle whose centre is at the point $(1, -2)$ and whose radius is 7.

6. Write the equation of the circle whose centre is in the position (p, q) and whose radius is $\sqrt{p^2 + q^2}$.

7. Write the equation of the circle whose centre is at the point $(0, 5)$ and which is tangent to the axis of X.

8. Write the equation of a circle passing through the origin and having its centre at the point $(3, 4)$.

9. Find the equation of a circle of which the line drawn from the origin to the point (p, q) shall be a diameter.

10. Find the equation of a circle of which the line from the point $(1, 3)$ to the point $(7, -5)$ shall be a diameter.

11. Find the locus of the centre of the circle passing through the points (p, q) and (p', q'), and show that it is a straight line perpendicular to the line joining these points.

Method of Solution. Since the two points are to lie on the circle, their co-ordinates must satisfy the equation of the circle; that is, we must have

$$(p - a)^2 + (q - b)^2 = r^2;$$
$$(p' - a)^2 + (q' - b)^2 = r^2.$$

The radius r being a quantity which must not appear in the equation, we must eliminate it, which we do by mere subtraction. We thus find an equation of the first degree between a and b, the co-ordinates of the centre. To express the locus in the usual form we may write x and y for a and b in this equation, which will then be the required equation of the locus of the centre.

12. Find the locus of the centre of the circle passing through the points (1, 1) and (7, 9).

13. Find the locus of the centre of the circle passing through the origin and the point (p, q).

14. Find the locus of the centre of the circle passing through the origin and the point $(2p \cos \alpha, 2p \sin \alpha)$.

66. *Intersections of Circles.* The points in which circles intersect each other, or in which a straight line intersects a circle, are found from the values of the co-ordinates which satisfy both equations.

Let the two circles which intersect be given by the equations

$$\left.\begin{aligned} x^2 + y^2 + ax + by + p &= 0; \\ x^2 + y^2 + a'x + b'y + p' &= 0. \end{aligned}\right\} \qquad (a)$$

By subtracting one of these equations from the other, we have

$$(a - a')x + (b - b')y + p - p' = 0;$$

whence
$$y = \frac{p' - p + (a' - a)x}{b - b'}.$$

By substituting this value of y in either of the equations (a), we shall have a quadratic equation in x.

Since such an equation has two roots, there will be two points of intersection.

But the roots may be imaginary. The circles will then not meet at all, but one will be wholly within or wholly without the other.

If the roots are equal, the points of intersection are coincident, and the circles touch each other.

EXERCISES.

1. Find the points of intersection and the length of the common chord of the two circles

$$x^2 + y^2 = r^2;$$
$$x^2 + y^2 - ax = d^2.$$

Ans. The co-ordinates are:

$$x = \frac{r^2 - d^2}{a} \text{ for both points;}$$

$$y = \pm \frac{1}{a} \sqrt{(ar + r^2 - d^2)(ar - r^2 + d^2)}.$$

$$\text{Common chord } = \frac{2}{a} \left\{ a^2 r^2 - (r^2 - d^2)^2 \right\}^{\frac{1}{2}}.$$

2. Find the points of intersection and the length of the common chord of the circles

$$x^2 + y^2 - a^2 = 0;$$
$$x^2 + y^2 + by - r^2 = 0.$$

3. Determine the radius r so that the circles

$$x^2 + y^2 = r^2,$$
$$x^2 + y^2 - 2x = 3,$$

shall touch each other.

Method of Solution. We find, as in the preceding exercises, the values of the co-ordinates x and y of the points of intersection. In order that the roots may be equal, the quantity under the radical sign in the expression for y must vanish. Equating it to zero, we shall have

$$r^4 - 10r^2 + 9 = 0,$$

an equation of which the roots are 3 and 1.

4. Find the distance apart of the two points in which the line

$$x = y + 1$$

intersects the circle

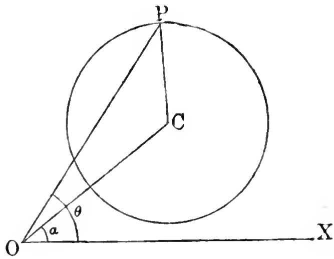
$$x^2 + y^2 = 10.$$

67. *Polar Equation to the Circle.*

Let O be the pole, OX the initial or base line;

ρ' and α the polar co-ordinates of the centre C;

ρ and θ the polar co-ordinates of any point P.



We then have, by trigonometry,

$$PC^2 = OP^2 + OC^2 - 2OP.OC \cos POC;$$

that is, $$r^2 = \rho^2 + \rho'^2 - 2\rho\rho' \cos(\theta - \alpha),$$

or $$\rho^2 - 2\rho\rho' \cos(\theta - \alpha) + \rho'^2 - r^2 = 0, \tag{6}$$

the polar equation required.

It may also be obtained from the equation referred to rectangular axes by putting $x = \rho \cos\theta$, $y = \rho \sin\theta$, $a = \rho' \cos\alpha$, and $b = \rho' \sin\alpha$. If the initial line pass through the centre, $\alpha = 0$ and the equation becomes

$$\rho^2 - 2\rho\rho' \cos\theta + \rho'^2 - r^2 = 0. \tag{7}$$

If the origin lie on the circumference, $\rho'^2 = r^2$ and the equation becomes

$$\rho = 2\rho' \cos\theta = 2r \cos\theta. \tag{8}$$

NOTE. In the above we put ρ and ρ' for the radii vectores in order to avoid confusing them with the radius of the circle, which we call r.

Tangents and Normals.

68. *Equation of Tangent to a Circle.* The requirement that a line shall be tangent to a circle does not alone determine the line, because a circle may have any number of tangents. We may therefore anticipate that this requirement will be expressed by an equation of condition between the parameters of the line. Let us then consider the problem:

To find the equation which the parameters of a line must satisfy in order that the line may be tangent to a given circle.

Let the circle be given by the equation

$$(x - a)^2 + (y - b)^2 = r^2,$$

and let the equation of the line be

$$Ax + By + C = 0.$$

By geometry, the situation of the line must be such that the perpendicular from the centre of the circle upon it shall be equal to the radius of the circle. *Conversely*, every line for which this perpendicular is equal to the radius of the circle is a tangent.

Now, the length of the perpendicular from the point (a, b) upon the line (A, B, C) is

$$-\frac{aA + bB + C}{\sqrt{A^2 + B^2}}.$$

The requirement that this perpendicular shall be equal to the radius r of the circle gives the equation

$$aA + bB + C = r\sqrt{A^2 + B^2}, \qquad (1)$$

which is the required equation of condition between the parameters A, B and C.

If the equation of the line is in the normal form

$$x \cos \alpha + y \sin \alpha - p = 0,$$

we shall have

$$\sqrt{A^2 + B^2} = \sqrt{\cos^2\alpha + \sin^2\alpha} = \pm 1,$$

and the equation (1) will assume the form

$$a \cos \alpha + b \sin \alpha - p = \pm r. \qquad (2)$$

69. *Equation of the tangent expressed in terms of the tangent of the angle which the line makes with the axis of X.*

Let $y = mx + b$ be the equation of the tangent, and

$$x^2 + y^2 = r^2$$

the equation of the circle. Eliminating y between these two equations, we have

$$(1 + m^2)x^2 + 2mbx + (b^2 - r^2) = 0,$$

which must have equal roots, since the tangent touches the circle in only one point. Now the condition that this equation may have equal roots is (§ 8)

$$m^2b^2 = (1 + m^2)(b^2 - r^2);$$

whence $$b = \pm r\sqrt{1 + m^2},$$

which substituted in the equation of the tangent gives

$$y = mx \pm r\sqrt{1 + m^2}. \qquad (3)$$

Conversely, every line whose equation is of this form is a tangent to the circle.

70. *Tangent determined by Two Conditions.* In the preceding article we employed only the one condition, that the line should be tangent to the circle. Hence the line could be completely found only when one of the parameters was given. In order to determine completely the tangent line, some other condition besides its tangency to the circle must be given. Examples of such conditions are:

That the tangent line shall touch the circle at a given point;

That it shall pass through a given point not on the circle;

That it shall also be tangent to a second circle.

71. PROBLEM. *To find the equation of the line tangent to a circle and passing through a given point.*

Let x' and y' be the co-ordinates of the given point, and

$$x \cos \alpha + y \sin \alpha - p = 0$$

the equation of the tangent. Since the tangent passes through the point (x', y'), we must have

$$x' \cos \alpha + y' \sin \alpha - p = 0, \tag{4}$$

which combined with (2) will determine the two parameters, α and p, of the line.

We may, however, first eliminate p by subtraction, which gives the equation

$$(a - x') \cos \alpha + (b - y') \sin \alpha = r.$$

The solution of this equation, which is obtained by methods given in trigonometry, will give the value of α. It may also be obtained algebraically by substituting for $\cos \alpha$ its equivalent, $\sqrt{1 - \sin^2 \alpha}$, or for $\sin \alpha$ its equivalent, $\sqrt{1 - \cos^2 \alpha}$, or for $\cos \alpha$ and $\sin \alpha$ their equivalents in terms of $\tan \alpha$, viz.,

$$\cos \alpha = \frac{1}{\sqrt{1 + \tan^2 \alpha}}; \qquad \sin \alpha = \frac{\tan \alpha}{\sqrt{1 + \tan^2}}.$$

In either case we shall have a quadratic equation, the unknown quantity in which will be either $\sin \alpha$, $\cos \alpha$ or $\tan \alpha$.

If we write, for brevity,

$$m \equiv a - x',$$
$$n \equiv b - y',$$

the solution of these equations will give

$$\left.\begin{aligned} \sin\alpha &= \frac{nr \pm m\sqrt{m^2+n^2-r^2}}{m^2+n^2}; \\ \cos\alpha &= \frac{mr \mp n\sqrt{m^2+n^2-r^2}}{m^2+n^2}; \\ \tan\alpha &= \frac{mn \pm r\sqrt{m^2+n^2-r^2}}{r^2-n^2}. \end{aligned}\right\} \tag{5}$$

The value of p by (4) is

$$p = x' \cos\alpha + y' \sin\alpha,$$

in which $\cos\alpha$ and $\sin\alpha$ must be replaced by their values given above. Substituting the values of $\cos\alpha$, $\sin\alpha$ and p in the equation

$$x\cos\alpha + y\sin\alpha - p = 0,$$

the result will be the required equation of the tangent passing through the point (x', y'). The double sign shows that there may be *two* tangents drawn to a circle from a point without it.

Case when the given point is on the circle. In this case we shall have

$$m^2 + n^2 - r^2 = 0,$$

and the values (5) of $\sin\alpha$ and $\cos\alpha$ become

$$\sin\alpha = \frac{n}{r};$$

$$\cos\alpha = \frac{m}{r};$$

and

$$p = \frac{mx' + ny'}{r}.$$

Substituting these values in the equation of the tangent, it becomes

$$mx + ny - mx' - ny' = 0,$$

or

$$m(x - x') + n(y - y') = 0.$$

If we substitute for m and n their values, the equation is

$$(a - x')(x - x') + (b - y')(y - y') = 0.$$

If we take the centre of the circle as the origin, we have

$$a = 0, \qquad b = 0;$$

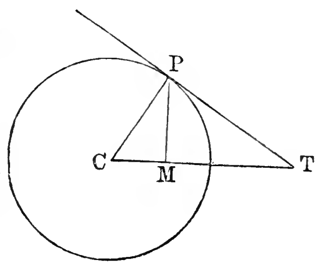
and because the point (x', y') is on the circle,

$$x'^2 + y'^2 = r^2.$$

Making these substitutions, the equation of the tangent assumes the simple form

$$x'x + y'y = r^2. \tag{6}$$

72. *Def.* The **subtangent** of a curve is the projection on the axis of X of that portion of the tangent intercepted between the point of tangency and its intersection with the axis of X.



Thus, if CT is the axis of X, MT is the subtangent corresponding to the tangent PT.

Length of the Subtangent. To find the length of MT, we find the intercept CT on the axis of X and subtract CM, the abscissa of the point of contact.

The equation of the tangent PT is (6)

$$x'x + y'y = r^2;$$

and when $y = 0$, we have

$$x = \frac{r^2}{x'} = CT.$$

Hence we have

$$MT = \frac{r^2}{x'} - x' = \frac{r^2 - x'^2}{x'}.$$

73. *Def.* The **normal** to any curve is the perpendicular to the tangent at the point of contact.

Equation of the Normal to a Circle. The equation of the line perpendicular to

$$x'x + y'y = r^2$$

and passing through the point of contact (x', y') is, by § 47,

$$y - y' = \frac{y'}{x'}(x - x'),$$

or

$$y'x - x'y = 0,$$

the equation required.

The form of this equation shows that every normal of a circle passes through the centre—a property which is easily established by elementary plane geometry.

The length of the normal is that portion of the line included between the point of contact and the axis on which the subtangent is measured. In the case of the circle, the normal is constant and equal to the radius.

Def. The **subnormal** of a curve is the projection of the normal on the axis of X.

Thus, CM is the subnormal corresponding to the point P, and in the circle is equal to the abscissa of the point of contact.

EXERCISES.

1. Show that the condition that the line

$$x \cos \alpha + y \sin \alpha - p = 0$$

shall be tangent to the circle

$$(x - a)^2 + (y - b)^2 = a^2 + b^2$$

is

$$a \cos \alpha + b \sin \alpha = p \pm \sqrt{a^2 + b^2}.$$

2. What is the condition that the line

$$y = mx + c \quad \times$$

shall be tangent to the circle $c = -(m+2) \pm \sqrt{4+(1+m^2)r^2}$

$$(x - 1)^2 + (y + 2)^2 = 16?$$

3. What must be the value of c in order that the equation

$$x + y = c$$

may be tangent to the same circle?

Ans. $c = -1 \pm 4\sqrt{2}$.

× In general the line $y = mx + b$ touches $(x-\alpha)^2+(y-\beta)^2=r^2$
if $b = -(m\alpha-\beta) \pm \sqrt{\beta^2+(1+m^2)r^2}$

4. What must be the value of m in order that the line

$$y = mx + 6$$

may be tangent to the circle

$$x^2 + y^2 = 16?$$

Ans. $m = \pm \frac{\sqrt{5}}{2}$.

5. In the last example show that we get the same answer for the line

$$y = mx - 6,$$

and explain the equality by geometric construction.

6. What must be the value of the radius d in order that the circle

$$(x + 3)^2 + (y - 1)^2 = d^2$$

may have as a tangent the line

$$y = 2x + 5?$$

Ans. $d = \frac{2}{\sqrt{5}}$.

7. By elementary geometry, two circles are tangent to each other when the distance of their centres is equal to the sum or difference of their radii. By means of this theorem write out the condition that the circles

$$(x - a)^2 + (y - b)^2 = r^2$$

and

$$(x - a')^2 + (y - b')^2 = r'^2$$

shall be tangent to each other.

8. Show that the length of the common chord of the circles whose equations are

$$(x - 2)^2 + (y - 3)^2 = 9$$

and

$$(x - 3)^2 + (y - 2)^2 = 9$$

is $\sqrt{34}$.

9. Find the condition that the circles

$$(x - h)^2 + (y - k)^2 = a^2$$

and

$$(x - k)^2 + (y - h)^2 = a^2$$

may touch each other.

Ans. $a = \frac{1}{\sqrt{2}}(k - h)$.

10. Show that the polar equation

$$\rho^2 - (a \cos \theta + b \sin \theta)\, \rho = p^2$$

is that of a circle, and express its radius and the position of its centre.

11. What curve does

$$\rho = a \cos(\theta - \alpha) + b \cos(\theta - \beta) + c \cos(\theta - \gamma) + \ldots$$

represent?

12. A point moves so that the sum of the squares of its distances from the four sides of a rectangle is constant. Show that the locus of the point is a circle.

13. Given the base of a triangle ($2b$) and the sum of the squares on its sides ($2m^2$), find the locus of the vertex when the middle point of the base is the origin.

Ans. $x^2 + y^2 = m^2 - b^2$.

14. Given the base (b) and the vertical angle (B) of a triangle, find the locus of the vertex when the origin is at the end of the base.

Ans. $x^2 + y^2 - bx - by \cot B = 0$.

15. Show that if, in the equation

$$x^2 + y^2 + Ax + By + C = 0,$$

we have

$$4C > A^2 + B^2,$$

the circle will be imaginary. It is enough to show that the radius is imaginary.

16. Show that a circle may be defined as the locus of a point the square of whose distance from a fixed point is proportional to its distance from a fixed line.

17. Show that a circle is the locus of a point the sum of the squares of whose distances from any number of fixed points is a constant.

18. If $\frac{a}{a'} = \frac{b}{b'}$, show that the circles

$$x^2 + y^2 + ax + by = 0,$$
$$x^2 + y^2 + a'x + b'y = 0,$$

touch at the origin.

19. Find the locus of a point whose distances from two fixed points have a fixed ratio to each other.

20. Express analytically the locus of a point from which a tangent drawn to a circle will have a fixed length t.

$y^2 = r^2 + t^2$

21. Find the locus of the point from which two adjoining segments of the same straight line shall be seen under the same varying angle. In other words, if A, B and C are three points in the same straight line, find the locus of the point X which will satisfy the condition

$$\text{Angle } AXB = \text{angle } BXC.$$

22. If the equation of the circle $x^2 + y^2 = r^2$ is transformed to another system of co-ordinates having the same axis but a different direction, show analytically that the equation will not be altered.

23. Show analytically that if a circle cuts out equal chords from the two co-ordinate axes, the co-ordinates a and b of its centre will be equal. Intercepts are $a \pm \sqrt{r^2 - a^2}$ & $b \pm \sqrt{r^2 - a^2}$

24. Find the equation of the circle which passes through the three fixed points (x_1y_1), (x_2y_2), (x_3y_3).

25. Having given the circle $x^2 + y^2 + 10x - 6y - 2 = 0$, find the equation of its two tangents, each of which is parallel to the straight line $y = 2x - 7$. $y = 2x + 13 \pm \sqrt{178}$

26. The circle $x^2 + y^2 = r^2$ has tangents touching it at the respective points $(x_1\ y_1)$ and $(x_2\ y_2)$. Express the tangent of the angle formed by these tangents. $\frac{x_1y_2 - x_2y_1}{x_1x_2 + y_1y_2}$

27. A line of fixed length slides along the axes of co-ordinates in such a way that one end constantly remains on each axis. What is the locus of the middle point of the line?

28. Given a point (a, b) and a finite straight line whose length is c, find the locus of the point whose distance from (a, b) is a mean proportional between c and its distance from the line $x \cos \alpha + y \sin \alpha = p$.

29. Having given the equation of the circle $y^2 = 2rx - x^2$, let chords be drawn from the origin to all points of the circle, and let each of them be divided in the constant ratio $m : n$. It is required to find the locus of the points of division.

30. The same thing being supposed, the chords, instead

of being divided, are each doubled. Find the locus of the ends.

31. On each radius of a circle having its centre at the origin a distance from the origin is measured equal to the ordinate of the terminal point of the radius on the circle. Find the locus of the point where the measures end. $x^2+y^2=ry$

32. The same thing being supposed, take on each radius a point at a distance from the centre equal to the abscissa of the end of the radius, and find the locus of this point. $x^2+y^2=rx$

33. Find the locus of the point from which two circles will subtend the same angle; that is, from which the angle subtended by the pair of tangents to one circle is equal to that subtended by the pair of tangents to the other circle. $\rho'=\rho\frac{r'}{r}$

34. Find the equation of that circle which passes through the origin and cuts off the respective intercepts p and q from the positive parts of the axes of X and Y. $(x-\frac{p}{2})^2+(y-\frac{q}{2})^2=r^2$

35. Find the locus of the point the sum of the squares of whose distances from the sides of an equilateral triangle is constant, and show that it is a circle. (To simplify the problem, let the base of the triangle be the axis of X.) $3(x^2+y^2)-a\sqrt{3}\,y+\frac{3}{4}a^2 =$

36. Find the polar equation of the circle when the origin is on the circumference and the initial line a tangent. $\rho=2r\sin\theta$

37. A line moves so that the sum of the perpendiculars AP and BQ from two fixed points, A and B, shall be a constant. Find the locus of the middle point of the segment PQ. $x^2+y^2=(\frac{p+q}{2})^2$

38. The straight line whose equation is $3y+5x+19=0$ cuts the circle $y^2+x^2=113$ in two points. What is the length of the chord which the circle cuts off from the line? r^2-p $=113-\frac{3}{3}$

39. Find the equation of the straight line which cuts the circle $x^2+y^2=169$ in two points whose abscissæ are respectively -12 and $+7$.

40. Find the equation of a line passing through the point (x', y') and forming in the circle $x^2+y^2=r^2$ a chord whose length is d. $x\cos\alpha+y\sin\alpha-p=0$ $p=\sqrt{r^2-\frac{d^2}{4}}$ $x'\cos\alpha+y'\sin\alpha-p=0$

41. Through the point (x', y'), inside the same circle, a chord is to be drawn which shall be bisected by the point. Find the equation of the chord.

$x\cos\alpha+y\sin\alpha-p=0$, $\sin\alpha=\frac{y'}{x'}\cos\alpha$

$p^2=x'^2+y'^2$

Systems of Circles.

74. Let us consider the expression

$$(x-a)^2+(y-b)^2-d^2,$$

which, for brevity, we shall represent by P, putting

$$P \equiv x^2+y^2-2ax-2by+a^2+b^2-d^2.$$

To every point on the plane will correspond a definite value of P, found by substituting the co-ordinates of such point in this value of P.

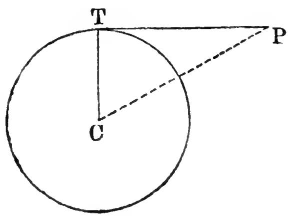
We may form any number of expressions of this form, such as

$$\begin{aligned} P' &\equiv x^2+y^2-2a'x-2b'y+a'^2+b'^2-d'^2; \\ P'' &\equiv x^2+y^2-2a''x-2b''y+a''^2+b''^2-d''^2; \\ &\quad \text{etc.} \qquad \text{etc.} \qquad \text{etc.} \end{aligned}$$

In general, the co-ordinates x and y which enter into P will be considered as entirely unrestricted, in which case P will be simply an algebraic function of x and y.

But we may also inquire about those special values of x and y which satisfy the equation $P=0$. We know from §§ 63, 64 that the points corresponding to these special values of x and y all lie on a circle of radius d, having its centre at the point (a, b). We now have the—

75. THEOREM. *The value of P for any point of the plane is equal to the square of the tangent from that point to the circle $P=0$.*



Proof. Let P be the point (x, y), and let C be the centre of the circle $P=0$, which is, by hypothesis, the point (a, b). We then have

$$CP^2=(x-a)^2+(y-b)^2;$$

and because PTC is a right angle,

$$\begin{aligned} PT^2 &= CP^2-CT^2 \\ &= (x-a)^2+(y-b)^2-d^2, \end{aligned}$$

which is the value of the function P, thus proving the theorem.

REMARK. If the point (x, y) is taken within the circle, P will be negative, and the length of the tangent, being the square root of P, will be imaginary.

76. THEOREM. *If $P = 0$ and $P' = 0$ are the equations of two circles, any equation of the form*

$$\mu P + \nu P' = 0 \tag{a}$$

will represent a third circle passing through their points of intersection.

Proof. We first show that (a) is the equation of *some* circle. Substituting for P and P' their values, we have for the equation of the curve

$$(\mu + \nu)(x^2 + y^2) - 2(\mu a + \nu a')x - 2(\mu b + \nu b')y \\ + \mu(a^2 + b^2 - d^2) + \nu(a'^2 + b'^2 - d'^2) = 0.$$

Here the coefficients of x^2 and y^2 are equal and there is no term in xy. Hence (§ 64) the curve represented by the equation (a) is some circle.

Secondly, the co-ordinates of all points in which the circles P and P' intersect must satisfy both of the equations $P = 0$ and $P' = 0$. Hence they also satisfy the equation

$$\mu P + \nu P' = 0,$$

and therefore the points of intersection lie on the circle of which the equation is (a).

Hence this circle passes through the points of intersection of the circles P and P'. Q. E. D.

Cor. The curve represented by (a) depends only on the *ratio* of the factors μ and ν, and remains unchanged when both are multiplied by the same quantity.

By assigning different values to the ratio $\mu : \nu$, we may determine as many circles as we please passing through two points.

A collection of circles passing through two points is called a **family of circles.**

77. PROBLEM. *To find the locus of the point from which the tangents to two circles shall have a given ratio to each other.*

Solution. Let $P = 0$ and $P' = 0$ be the equations of the two circles, and let the tangents from the moving point be in the ratio $m : m'$.

The square of the tangents will then be in the ratio $m^2 : m'^2$. But these squares are represented by the respective values of P and P' corresponding to the point from which the tangents are drawn. Hence between these values of P and P' we have the proportion

$$P : P' = m^2 : m'^2,$$

which gives

$$m'^2 P - m^2 P' = 0. \qquad (b)$$

Because the co-ordinates of the point from which the tangents are drawn must satisfy this equation, this equation is that of the required locus.

Comparing with § 76, we see that the equation is of the form (a). Hence:

THEOREM. *The locus of the point from which the lengths of the tangents drawn to two circles have a constant ratio to each other is a third circle, passing through the common points of intersection of the first two circles, and therefore a third circle of the same family.*

78. *The Radical Axis.* If the ratio $m : m'$ is unity, the equation (b) will reduce to

$$P - P' = 0,$$

or, substituting for P and P' their values,

$$2(a' - a)x + 2(b' - b)y + a'^2 - a^2 + b'^2 - b^2 - d'^2 + d^2 = 0,$$

which, being of the first degree, is the equation of a straight line. From the results of § 76, this line must be the common chord of the two circles. Hence:

THEOREM. *The locus of the point from which the tangents to two circles are equal is the common chord of the two circles.*

This locus is called the **radical axis** of the two circles.

Imaginary Points of Intersection.

79. The theorem of § 77 holds equally true whether the circles P and P' intersect or not; that is, it leads to a third circle passing through the points of intersection of two circles, *even when these two circles do not intersect.* If in this last case the third circle, which we may call P'', really intersected either of the others, say P', this result would be self-contradictory. For in such a case the circle P could not pass through the intersection of P' and P'', and so the result of the theorem would be false.

But if the point of intersection of P and P' is *nowhere*, there will be nothing contradictory in the result, if only P'' intersects each of them *nowhere.*

Again, in § 78 we have found a perfectly general equation of the radical axis founded on the definition that the radical axis is the line joining the points of intersection of two circles, which equation gives the real radical axis even when the circles do not intersect.

If we take any special case, we shall find that the algebraic processes are the same whether the circles do or do not really intersect: only, in the latter case, the co-ordinates of the points of intersection will be imaginary. To illustrate this in the simplest way, take the two circles

$$x^2 + y^2 = 1;$$
$$(x - 3)^2 + (y - 3)^2 = 9.$$

To determine the points of intersection we must find values of x and y which satisfy both equations. The second equation is, by reduction,

$$x^2 + y^2 - 6x - 6y + 18 = 9. \qquad (a)$$

Substituting the value of $x^2 + y^2 = 1$ in the first member, we find

$$x + y = \frac{10}{6} = \frac{5}{3}. \qquad (b)$$

Hence

$$y = \frac{5}{3} - x; \qquad y^2 = x^2 - \frac{10}{3}x + \frac{25}{9}.$$

By substituting for y^2 its value $1 - x^2$, we find

$$2x^2 - \frac{10}{3} x = 1 - \frac{25}{9} = - \frac{16}{9};$$

$$x^2 - \frac{5}{3} x = - \frac{8}{9}.$$

Completing square,

$$x^2 - \frac{5}{3} x + \frac{25}{36} = \frac{25 - 32}{36} = - \frac{7}{36}.$$

The square being negative shows that the roots are imaginary. The solution gives, for the points of intersection,

$$x = \frac{5 \pm \sqrt{-7}}{6};$$

$$y = \frac{5 \mp \sqrt{-7}}{6}.$$

The co-ordinates x and y being imaginary, the circles do not really intersect. But these imaginary values of the co-ordinates satisfy the equations of both circles and also the equation (b) of the radical axis, as we readily find by the calculation:

$$x^2 = \frac{5^2 - 7 \pm 10 \sqrt{-7}}{36} = \frac{9 \pm 5 \sqrt{-7}}{18};$$

$$y^2 = \frac{5^2 - 7 \mp 10 \sqrt{-7}}{36} = \frac{9 \mp 5 \sqrt{-7}}{18};$$

$$x + y = \frac{5 \ \pm 7 + 5 \mp 7}{6} = \frac{10}{6} = \frac{5}{3}$$

In taking the sum of the first two equations, the imaginary terms cancel each other and we have $x^2 + y^2 = 1$.

Subtracting 6 times the third equation we satisfy (a), and the third is identical with (b), which is the equation of the radical axis.

We adopt the following forms of language to meet this class of cases:

I. An **imaginary point** is a fictitious point which we *suppose* or *imagine* to be represented by imaginary co-ordinates.

II. When imaginary co-ordinates satisfy the equation of a

curve, we may talk about the corresponding imaginary points as belonging to that curve.

III. A curve may be entirely imaginary.

Example. The equation

$$x^2 + y^2 - 2x - 2y = \cancel{1} - 3$$

is that of a circle. But we may write it in the form

$$(x - 1)^2 + (y - 1)^2 = - 1.$$

The first member is a sum of two squares, and therefore positive for all real values of x and y, while the second member is negative. Hence there are no real points whose co-ordinates satisfy the equation.

EXERCISES.

1. If we take, on the axis of X, two imaginary points whose abscissas are $a + bi$ and $a - bi$ respectively, find the abscissa of the middle point between them. $i =$

2. Using the method of § 45, find the equation of the line joining the imaginary points whose co-ordinates are—

$$\begin{array}{ll} \text{1st point: } x' = ci, & y' = a + 2ci; \\ \text{2d point: } x'' = b + ci, & y'' = a + 2b + 2ci; \end{array}$$

and show that it is the real line

$$y = 2x + a.$$

3. Find the equation of the circle whose centre is at the point $(a, 2b)$ and which cuts the axis of X at the points described in Ex. 1.

Ans. $(x - a)^2 + (y - 2b)^2 = 3b^2.$

4. Find the equation of a circle belonging to the family fixed by the pair

$$(x - 4)^2 + (y - 3)^2 = 16,$$
$$(x - 2)^2 + (y - 5)^2 = 9,$$

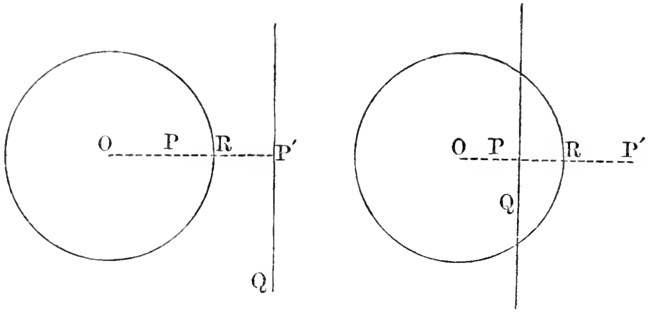
and passing through the origin.

Ans. $\mu = 20$; $\nu = -9$; Eq.: $11(x^2 + y^2) - 124x - 30y = 0.$

Section II. Synthetic Geometry of the Circle.

Poles and Polars.

80. Let there be two points, P and P', on the same straight line from the centre O of a fixed circle, and so situ-



ated that the radius OR shall be a mean proportional between OP and OP'.

Through either of the points, as P', draw a line Q perpendicular to the radius. Then

The line Q is called the **polar** of the point P with respect to the circle, and the point P is called the **pole** of the line Q with respect to the circle.

Had we drawn the line through P, it would have been the *polar* of the point P', and P' would have been the *pole* of the line through P.

81. The following propositions respecting poles and polars flow from these definitions:

I. To every point in the plane of the circle corresponds *one* definite polar, and to every line *one* definite pole.

II. The polar of a point and the pole of a line may be found by construction as follows:

(a) If the pole P is given, we draw the radius through the pole intersecting the circle at R. We then find the point P' by the proportion $OP : OR = OR : OP'$.

The perpendicular through P' will be the polar of P.

(b) If the polar is given, we draw the perpendicular from

the centre O upon the polar, and produce it if necessary. If it intersects the circle at R and the polar at P', we determine OP as the third proportional to OP' and OR. The point P will then be the required pole.

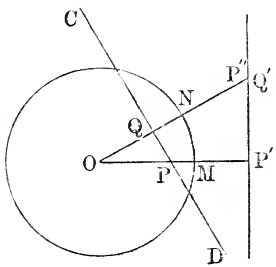
III. When the pole is within the circle, the polar is wholly without it.

IV. If a pole is without the circle, the polar cuts the circle.

V. When the pole is a point on the circle, the polar is the tangent at that point.

VI. If the pole approaches indefinitely near the centre of the circle, the polar recedes indefinitely, and *vice versa*.

82. Fundamental Theorem. *If a line pass through a point, the polar of the point will pass through the pole of the line.*



Proof. Let the line CD pass through the point P.

By definition, we find the polar of P by drawing the radius OM through P, taking the point P' so that, putting r for the radius OM,

$$OP : r = r : OP', \qquad (a)$$

and drawing $P'Q' \perp OP'$.

We find the pole of CD by drawing $OQ \perp CD$ and finding a point P'' such that

$$OQ : r = r : OP''. \qquad (b)$$

We have to prove that P'' lies on the polar $P'Q'$. If we call Q' the point in which OQ meets the polar $P'Q'$, the triangles $P'OQ'$ and QOP, being both right-angled and having the angle at O common, are equiangular and therefore similar. Hence

$$OQ : OP = OP' : OQ'. \qquad (c)$$

Comparing the proportions (a) and (b), we have

$$OP \,.\, OP' = OQ \,.\, OP'',$$

which gives the proportion

$$OQ : OP = OP' : OP''.$$

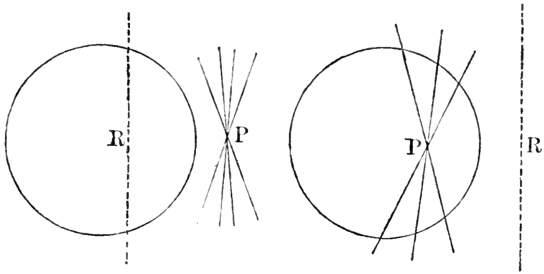
Comparing this proportion with (c), we have

$$OQ' = OP''.$$

Hence P'' and Q' coincide; that is, the pole P'' lies on the polar $P'Q'$. Q. E. D.

Cor. 1. We may imagine several lines all passing, like CD, through the point P. The theorem shows that the poles of these lines all lie on $P'Q'$. Hence,

If several lines pass through a point, their poles will all lie upon the polar of the point.



Cor. 2. We may imagine several points, all lying, like P, on the line CD. The theorem shows that the polars of these points will all pass through Q', the pole of CD. Hence,

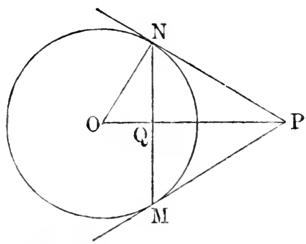
If several points lie in a straight line, their polars will all pass through the pole of the line.

REMARK. These several theorems may be more readily grasped when placed in the following form:

1. *If a line turn round on a point, its pole will move along the polar of that point.*

2. *If a point move along a line, its polar will turn round on the pole of that line.*

83. THEOREM I. *If from any point two tangents be drawn to a circle, the line joining the points of contact will be the polar of the point.*



Proof. Let the tangents from P touch the circle at M and N. Let Q be the point in which OP, from the centre O, intersects the line MN.

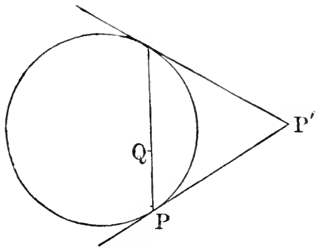
By elementary geometry, OQN and ONP are right triangles. Because they have the angle at O common, they are equiangular and similar. Hence

$$OQ : ON = ON : OP.$$

Now, since ON is the radius of the circle, this proportion shows that MN is the polar of P. Q. E. D.

THEOREM II. *If through any point a chord be drawn to a circle, the tangents at the extremities of the chord will meet on the polar of the point.*

Proof. Let the chord pass through the point Q, and let the tangents meet at P'. By Theorem I., P' is the pole of the chord; therefore, because Q lies on the chord, the polar of Q passes through P', the pole of the chord. Q. E. D.



Cor. 1. *If any number of chords be drawn through the same point, the locus of the point in which the tangents at their extremities intersect will be a straight line, the polar of the point.*

Cor. 2. Conversely, *If from a moving point on a straight line tangents be drawn to a fixed circle, the chords joining the corresponding points of tangency will all pass through the pole of the line.*

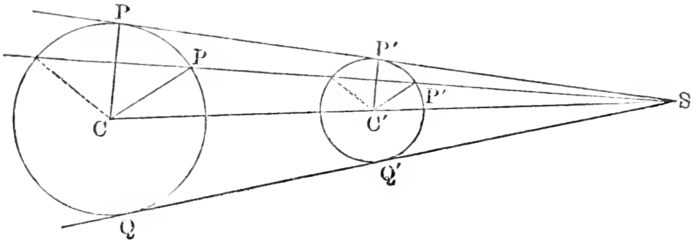
THEOREMS FOR EXERCISE.

1. If we take any four points, A, B, A' and B', on a circle, and if P be the point of meeting of the tangents at A and B, and P' the point of meeting of the tangents at A' and B', then the point of meeting of the lines AB and $A'B'$ will be the pole of PP'.

2. If we take four points, A, B, X and Y, on a circle, such that the tangents at A and B and the secant XY pass through a point, then the tangents at X and Y and the secant AB will also pass through a point.

Centres of Similitude.

84. *Def.* The line joining the centres of two circles is called their **central line.**



THEOREM. *If the ends of parallel radii of two circles be joined by straight lines, these lines will all pass through a common point on the central line.*

Proof. Let CP and $C'P'$ be any two parallel radii, and S the point in which the line PP' intersects the line CC' joining the centres. The similar triangles SPC, $SP'C'$ give the proportion

$$SC : SC' = CP : C'P'. \qquad (a)$$

Putting, for brevity, $r =$ the radius CP, and $r' =$ the radius $C'P'$, this proportion gives, by division,

$$SC - SC' : SC' = CP - C'P' : C'P' = r - r' : r',$$

or
$$CC' : SC' = r - r' : r'.$$

Hence
$$SC' = \frac{r'}{r - r'} CC';$$

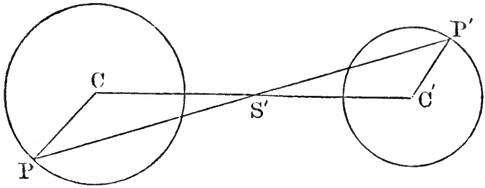
that is, the distance SC' is equal to the line CC' multiplied by a factor which is independent of the direction of the radii CP, $C'P$; therefore the point S is the same for all pairs of parallel radii. Q. E. D.

Case of oppositely directed radii. If the radii CP, $C'P'$ be drawn in opposite directions, it may be shown in a similar

way that the line PP' intersects the central line CC' in a point S' determined by the proportion

$$CS' : C'S' = CP : C'P' = r : r', \qquad (b)$$

whence S' is a fixed point in this case also.



Def. The two points through which pass all lines joining the ends of parallel radii of two circles are called the **centres of similitude** of the two circles.

The **direct centre of similitude** is that determined by similarly directed radii.

The **inverse centre of similitude** is that determined by oppositely directed radii.

Corollaries. The following corollaries should, so far as necessary, be demonstrated by the student.

I. *The direct centre of similitude is always without the central line of the two circles, and the inverse centre is always within this line, however the two circles may be situated.*

II. *If the two circles are entirely external to each other, the centres of similitude are the points of meeting of the pairs of common tangents to the two circles.*

III. Comparing the proportions (a) and (b), we see that the point S divides the line CC' externally into segments having the ratio $r : r'$, while S' divides it internally into segments having this same ratio.

This is the definition of a harmonic division. Hence

The two centres of similitude divide harmonically the line joining the centres of the two circles.

85. The following are fundamental theorems relating to centres of similitude:

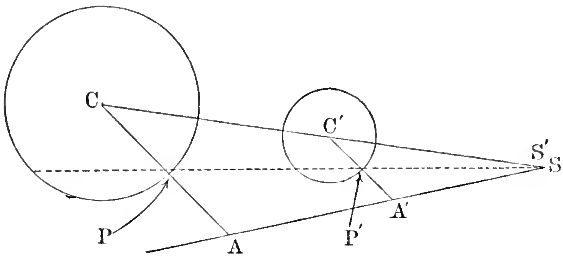
THEOREM I. *Every line similarly dividing two parallel radii of two circles passes through their centre of similitude.*

Proof. Let A and A' be the points in which the line divides the radii;

S', the point in which this line cuts the central line;

S, the centre of similitude;

r, r', the radii of the circles.



We then have, by the property of the centre of similitude,

$$C'S : CC' = C'P' : CP - C'P' = r' : r - r'. \qquad (a)$$

The similar triangles CAS' and $C'A'S'$ give

$$C'S' : CS' = C'A' : CA;$$

whence, by division,

$$C'S' : CC' = C'A' : CA - C'A'. \qquad (b)$$

By hypothesis, the radii are similarly divided at A and A';

$$\therefore \qquad C'A' : CA = r' : r;$$

whence, by division,

$$C'A' : CA - C'A' = r : r - r'.$$

Comparing with (a) and (b),

$$C'S' : CC' = C'S : CC';$$

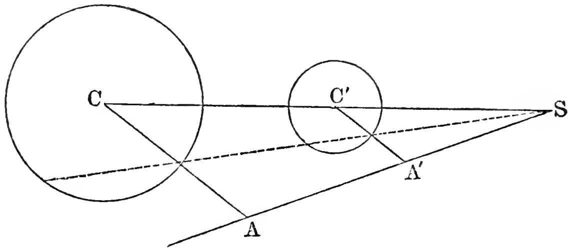
whence $C'S' = C'S$ and the points S and S' coincide. Q.E.D.

REMARK 1. If the radii are oppositely directed, the centre of similitude will be the inverse one. The demonstration is the same in principle.

REMARK 2. The demonstration may be shortened by employing the theorem of geometry that there is only one

point, internal or external, in which a line can be divided in a given ratio. (See Elementary Geometry.) By the fundamental property of the centre of similitude, it divides the central line into segments proportional to the radii of the circles. It may be shown that the point S' divides the central line into segments proportional to CA and $C'A'$. From this the student may frame the demonstration as an exercise.

Theorem II. Conversely, *If any line pass through a centre of similitude, and parallels be drawn from the centres of the circles to this line, the lengths of these parallels will be proportional to the radii of the circles.*



The demonstration is so easy that it may be supplied by the student.

86. *The Four Axes of Similitude of Three Circles.* If there be three circles, they form three pairs, each with its direct and inverse centre of similitude. There will therefore be six such centres in all, three direct ones and three inverse ones. The following propositions relate to this case:

Theorem III. *The three direct centres of similitude lie in a straight line.*

Proof. Let r_1, r_2 and r_3 be the radii of the three circles.

Let S_1, S_2 and S_3 be the direct centres of similitude of the pairs of circles (2, 3), (3, 1), (1, 2) respectively.

Let a line AB be passed through S_1 and S_2.

From the centres of the three circles draw three parallel lines, R_1, R_2 and R_3 to the line AB. Then,

Because AB is a line passing through the centre of similitude S_1,

$$R_2 : R_3 = r_2 : r_3. \qquad \text{(Th. II.)}$$

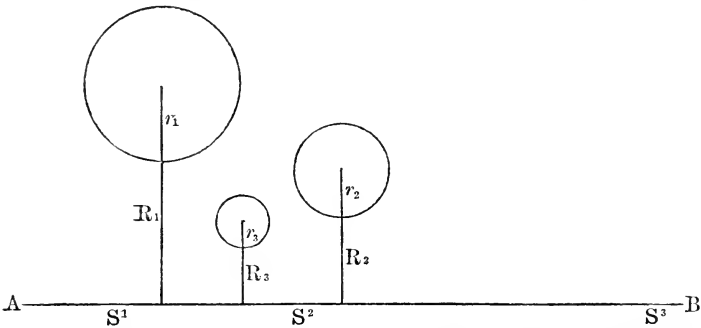
Because AB is a line passing through S_2,

$$R_1 : R_3 = r_1 : r_3.$$

Taking the quotients of these ratios, we have

$$R_1 : R_2 = r_1 : r_2;$$

hence the line AB divides the radii r_1 and r_2 similarly.



Therefore this line passes through the centre of similitude S_3 of the circles (1, 2). (Th. I.) Q. E. D.

THEOREM IV. *Each direct centre of similitude lies in the same line with the two inverse centres of similitude which are not paired with it.*

The demonstration of this theorem is so nearly like that of the last one that it may be supplied by the student.

Def. A straight line which contains three centres of similitude of a system of three circles is called an **axis of similitude** of the system.

Corollary. For each system of three circles there are four axes of similitude, of which one contains the three direct centres of similitude, and the others each contain one direct and two inverse centres.

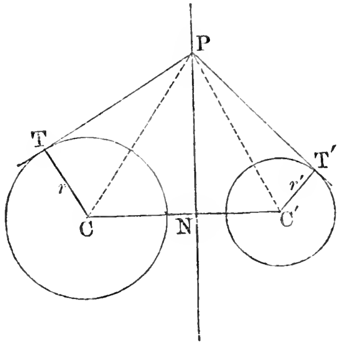
EXERCISES.

1. Show that if two of the three circles be equal, two of the axes of similitude will be parallel, and *vice versa.*

2. If all three circles are equal, describe the axes of similitude.

The Radical Axis.

87. Theorem. *If any perpendicular be drawn to the central line of two circles, the difference of the squares of the tangents from any one point of this perpendicular will be the same as from every other point of it.*



Proof. Let PN be any perpendicular to the central line of the circles C and C', and P any point on this perpendicular.

Let R and R' represent the distances of P from C and C'.

Because the tangents PT and PT' meet the radii drawn to the points T and T' of contact at right angles, we have

$$PT^2 = R^2 - r^2;$$
$$PT'^2 = R'^2 - r'^2.$$

Hence, for the difference of the squares of the tangents,

$$PT^2 - PT'^2 = R^2 - R'^2 - (r^2 - r'^2). \qquad (1)$$

From the right triangles PNC and PNC', we find, in the same way,

$$R^2 - R'^2 = NC^2 - NC'^2;$$

whence, from (1),

$$PT^2 - PT'^2 = NC^2 - NC'^2 - (r^2 - r'^2). \qquad (2)$$

The second member of this equation has the same value at whatever point on the perpendicular P may be situated, which proves the theorem.

Corollary. If we choose the point N so as to fulfil the condition

$$NC^2 - NC'^2 = r^2 - r'^2, \qquad (3).$$

we shall have

$$PT^2 = PT'^2,$$

and the tangents will be equal from every point of the perpendicular, which will then, by definition, be the radical axis.

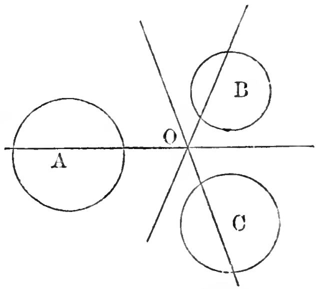
88. *Case when the circles intersect.* In this case the tangents drawn from either point of intersection are both zero and therefore equal. Hence this point is on the radical axis, and this axis is then the common chord (or secant) of the two circles. Hence another definition:

The radical axis of two circles is their common chord, produced indefinitely in both directions.

EXERCISE.

In the preceding construction the circles have been drawn completely outside of each other. Let the student extend the general proof (1) to the case when the circles intersect, showing that the two tangents from every point of the common secant are equal, and (2) to the case when one circle is wholly within the other, showing that the radical axis is then wholly without the outer circle.

89. *The Radical Centre of Three Circles.* If we have three circles, each of the three pairs will have its radical axis. We now have the theorem:



The three radical axes of three circles intersect in a point.

Proof. Let A, B and C be the three circles, and let O be the point in which the radical axis of A and B intersects the radical axis of B and C.

Because O is on the radical axis of A and B,

$$\text{Tangent } O \text{ to } A = \text{tangent } O \text{ to } B.$$

Because O is on the radical axis of B and C,

$$\text{Tangent } O \text{ to } B = \text{tangent } O \text{ to } C.$$

Hence $\quad \text{Tangent } O \text{ to } A = \text{tangent } O \text{ to } C;$

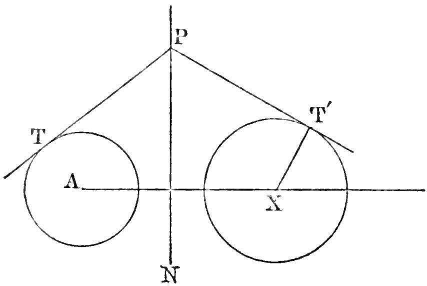
whence O lies on the radical axis of A and C, and all three axes pass through O.

Def. The point in which the three radical axes intersect is called the **radical centre** of the three circles.

Cor. The radical centre of three circles is a certain point from which the tangents to the three circles are all equal.

90. *System of Circles having a Common Radical Axis.* The theory of a family of circles, developed analytically in the preceding section, will now be explained synthetically.

PROBLEM. *Let us have a circle A and a straight line N: it is required to find a second circle X, such that N shall be the radical axis of the circles A and X.*



Solution. From the centre A draw an indefinite line AX perpendicular to the line N.

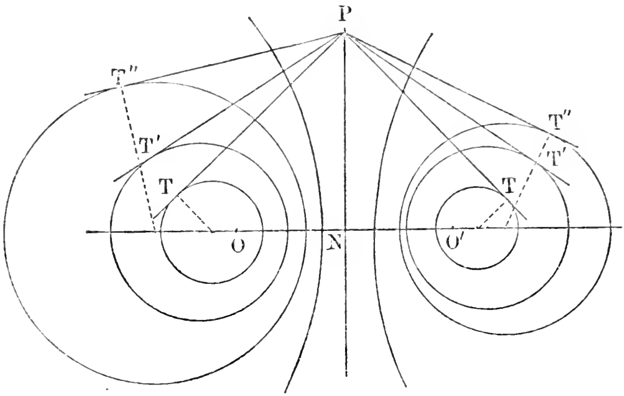
Take any point P on the radical axis N, and from it draw a tangent PT to the given circle.

From the same point, P, draw another line, PT', in any direction whatever; make $PT' = PT$, and from T' draw $T'X$ perpendicular to PT' and meeting the central line in X. The circle round the centre X with the radius XT' will be that required.

For PT', being perpendicular to the radius, is tangent to the circle X; and because $PT = PT'$, the line through P perpendicular to the central line is the radical axis. Hence the given line N is the radical axis of the two circles; whence the circle X fulfils the condition of the problem.

Since the line PT' may be drawn in any direction whatever, we may find an indefinite number of circles which fulfil the conditions of the problem.

The construction of these circles is shown in the figure. Since the tangents from P are all equal, it follows that the



line PN is the radical axis of any two circles of a family passing through the same two points, real or imaginary.

Tangent Circles.

91. The following propositions lead to the solution of the noted problem of drawing a circle tangent to three given circles.

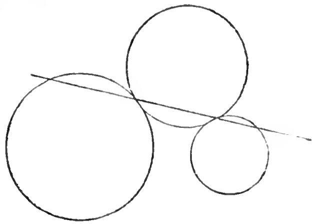
Def. When two circles each touch a third, the line through the points of tangency is called the **chord of contact.**

When two circles touch each other, either one must be wholly within the other, or each must be wholly without the other. Hence contacts are said to be of two kinds, *internal* and *external.*

THEOREM I. First, *If a circle is tangent to a pair of other circles, the chord of contact passes through a centre of similitude of the pair.*

Secondly, *This centre of similitude is the direct one when*

the contacts are of the same kind, and the inverse one when they are of opposite kinds.



Proof. The points of contact are readily shown to be centres of similitude of the respective pairs of tangent circles.

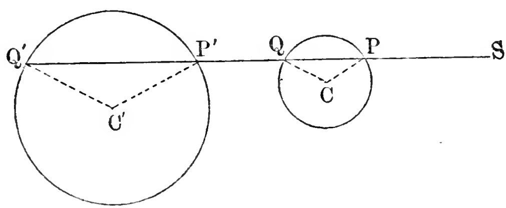
By § 86, any two centres of similitude of different pairs lie on a straight line with one of the centres of similitude of the third pair.

Hence the points of tangency are in the same line with a centre of similitude. Q. E. D.

Remark 1. An independent proof of the theorem is obtained by drawing the radii from each centre of the pair of circles to the points in which the joining line intersects the circumferences, and showing that the radii, taken two and two, are parallel.

Remark 2. The second part of the theorem is left as an exercise for the student.

92. *Homologous Points.* If a common secant to two circles be drawn through either of their centres of similitude, it will intersect each circle in two points. By combining

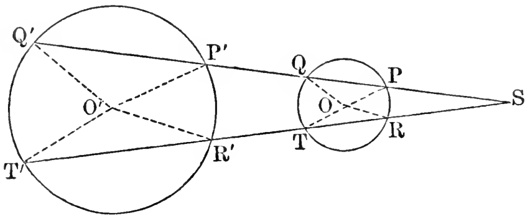


either of these points on one circle with either of the points on the other circle we may form four pairs of points, as (P, P'), (Q, Q'), (Q, P'), and (P, Q'). The pairs at the termini of parallel radii, namely, (P, P') and (Q, Q'), are called **homologous points;** those at the termini of non-parallel radii, as (Q, P') and (P, Q'), are called **anti-homologous.**

93. THEOREM II. *If two secants be drawn through a centre of similitude, then—*

I. *The distances of any two homologous points on one secant from the centre of similitude will be proportional to the distances of the corresponding points on the other secant.*

II. *The products of the distances of two anti-homologous points will be the same on the two secants.*



Hypothesis. Two secants, SQ' and ST', from the centre of similitude S, cut the circles in the points P, Q, P' and Q', and R, T, R' and T' respectively.

Conclusions:

$$\text{I. } SP : SP' = SR : SR';$$
$$SQ : SQ' = ST : ST'.$$
$$\text{II. } SP \,.\, SQ' = SR \,.\, ST' = SQ \,.\, SP' = ST \,.\, SR'.$$

Proof. I. Draw the central line and the radii to the points of intersection. Because of the parallelism of the radii OP and OP', etc., we have

Triangle SOP similar to triangle $SO'P'$;
Triangle SOQ similar to triangle $SO'Q'$;
Triangle SOR similar to triangle $SO'R'$;
Triangle SOT similar to triangle $SO'T'$.

From the similarity of these triangles, we have

$$SO : SO' = SP : SP' = SQ : SQ'$$
$$= SR : SR' = ST : ST'. \qquad \text{Q. E. D.}$$

II. The second and last of these proportions give

$$\left.\begin{aligned} SP \,.\, SQ' &= SP' \,.\, SQ; \\ SR \,.\, ST' &= SR' \,.\, ST. \end{aligned}\right\} \qquad (a)$$

By a fundamental property of the circle, shown in elementary geometry,

$$SP \cdot SQ = SR \cdot ST;$$
$$SP' \cdot SQ' = SR' \cdot ST'.$$

Multiplying these equations, we have

$$SP \cdot SQ' \times SP' \cdot SQ = SR \cdot ST' \times SR' \cdot ST.$$

By substitution from (a), this equation becomes

$$(SP \cdot SQ')^2 = (SR \cdot ST')^2;$$

whence, extracting the square root and combining with (a), we have conclusion II. Q. E. D.

94. *Def.* When a circle touches two others, we call it a *direct tangency* when the two tangencies are of the same kind, and an *inverse tangency* when they are of opposite kinds.

Several pairs of tangencies, all direct or all inverse, may be called *of the same nature.* If one pair is direct and another inverse, they are of *opposite natures.*

REMARK. It will be noted that the chords of contact pass through the same centre of similitude in the case of two pairs of tangencies of the same nature, but not otherwise. Hence, in what follows, whenever we have several circles touching two others, we shall suppose the tangencies to be of the same nature.

95. THEOREM III. *If each circle of one pair is a tangent of the same nature to the two circles of another pair, then the radical axis of each pair passes through a centre of similitude of the other pair.*

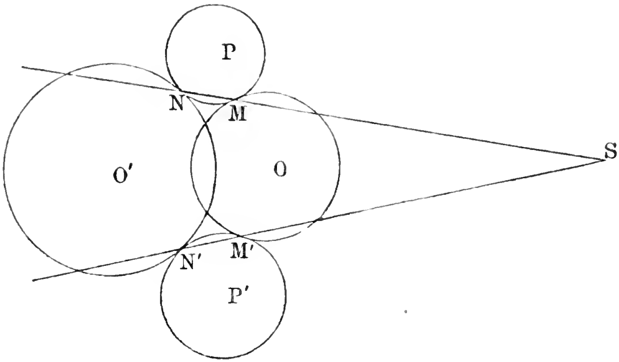
Proof. Let the circles P and P' touch the circles O and O' at the points M, N, M' and N'.

The point of meeting, S, of the lines NM and $N'M'$ will be a centre of similitude of O and O' (§ 91). Hence we have

$$SM \cdot SN = SM' \cdot SN'. \qquad (\S\ 93)$$

But $SM \cdot SN$ is equal to the square of the tangent from S to the circle P (El. Geom.), and $SM' \cdot SN'$ is the square of the

tangent from S to the circle P'. The tangents being equal, S is on the radical axis of P and P'. Q. E. D.



It is shown in the same way that a centre of similitude of P and P' is on the radical axis of O and O'. Q. E. D.

Cor. 1. If each of three circles is a tangent of the same nature to two other circles, then, by this theorem, one of the centres of similitude of each two out of the three circles must lie on the radical axis of the two circles which they touch. Hence,

When each of two circles touches each of three other circles, their radical axis will form one of the axes of similitude of the three circles.

Cor. 2. The same thing being supposed, each radical axis of the three circles will, by the theorem, pass through a centre of similitude of the pair which they touch. This centre of similitude will therefore be their point of intersection. Hence,

When each of two circles touches each of three other circles, the radical centre of the three circles will be a centre of similitude of the two circles.

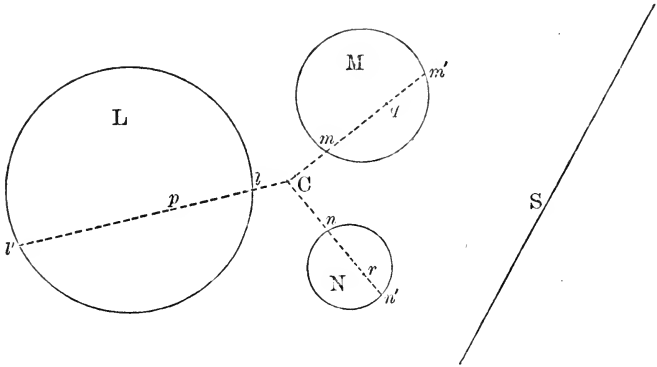
96. Problem. *To draw a circle tangent to three given circles.*

Construction. Let L, M and N be the three circles.

Find their radical centre, C, and an axis of similitude, S.

Find the poles p, q, r, of S with respect to the three circles.

Join Cp, Cq and Cr, and let l, m, n and l', m', n' be the points in which these lines intersect the three circles.



The circle through the three points l, m, n will be one of the tangent circles required, and the circle through the three points l', m', n' will be the other.

Proof. The axis of similitude S is the radical axis of *some* pair of circles touching the three given circles (§ 95, *Cor.* 1), and C is one of their centres of similitude (§ 95, *Cor.* 2).

Let us call X and Y the two circles of this pair, which are not represented in the figure.*

Let m, l, n and m', n', l', instead of being defined by the above construction, be defined as the points of tangency of this pair of circles whose centre of similitude is at C. Then, by (§ 91), the lines mm', nn' and ll' will all pass through C.

Through m' draw the common tangent to the circles M and X, and through m draw the common tangent to M and Y, and let P be the point of meeting of these tangents.

* The tangent circles and tangents are omitted from the printed figure to avoid confusing it. The student can supply them so far as necessary.

Then, because the tangents Pm and Pm' touch the same circle M at m and m', they are equal.

Hence these lines are also equal tangents to the circles X and Y; hence P lies on the radical axis of X and Y, that is, on the line S (§ [redacted]).

Because the line mm' is the chord of contact of tangents from P, it is the polar of P; hence the pole of S, a line through X, lies on the polar mm'. That is, the line Cq, found by the construction, passes through the points of contact m and m'.

In the same way it is shown that the points of contact l, l' and n, n' are upon the lines joining C and the poles p and r. Q. E. D.

CHAPTER V.

THE PARABOLA.

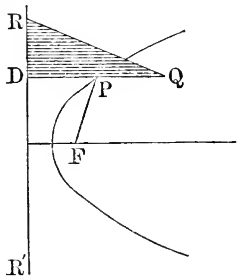
Equation of the Parabola.

97. *Def.* A **parabola** is the locus of a point which moves in a plane in such a way that its distances from a fixed point and from a fixed straight line in that plane are equal.

The fixed point is called the **focus,** and the fixed straight line the **directrix** of the parabola.

The curve is traced mechanically as follows:

Let F be the fixed point or focus, and RR' the fixed straight line or directrix. Along the latter place the edge of a ruler, and to the focus attach one end of a thread whose length is equal to that of a second ruler, DQ, right-angled at D. Then having attached the other end of the thread to the ruler at Q, stretch the thread tightly against the edge of the ruler DQ with the point of a pencil, while the ruler is moved on its edge DR along the directrix RR': the path of P will be a parabola. For in every position we shall have

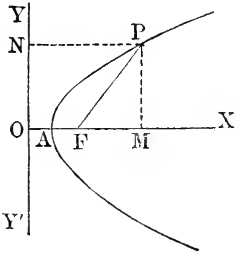


$$PF = PD,$$

which agrees with the definition.

98. Problem. *To find the equation of the parabola.*

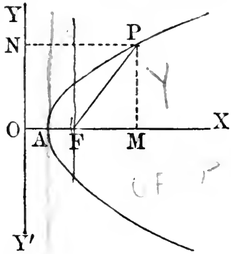
Let F be the focus, and YY' the directrix. Through F draw OX perpendicular to YY'. Take O as the origin, OF as the axis of X, and the directrix OY as the axis of Y. Put $OF = p$, and let P be any point on the curve. Join PF, and draw PN perpendicular to the directrix YY'. Then, by the definition of the curve, we have



$$PF = PN.$$

Let OM, PM, the co-ordinates of P, be x and y. Then we have

$$PM^2 + FM^2 = PF^2$$
$$= PN^2$$
$$= OM^2;$$



that is, $y^2 + (x - p)^2 = x^2$,

or $y^2 = 2p\,(x - \frac{1}{2}p)$, (1)

which is the equation of the parabola with the assumed origin and axes.

When $y = 0$ we have $x = \frac{1}{2}p$; that is, $OA = AF$, or the curve bisects the perpendicular distance between the focus and the directrix; and since there is no limit to the possible distance of a point from both focus and directrix, the curve extends out to *infinity*. From (1) we see that for every positive value of x greater than p there are *two* values of y, equal in magnitude but of opposite signs. Hence the curve is symmetrical with respect to the axis of X. If x be *negative* or *less* than $\frac{1}{2}p$, the values of y are imaginary; therefore no part of the curve lies to the left of A.

Def. The point A where the curve intersects the perpendicular from the focus on the directrix is called the **vertex** and AX the **axis** of the parabola.

The equation (1) will assume a simpler and more useful form by transferring the origin to the vertex, which is done by simply writing x for $x - \frac{1}{2}p$; hence (1) becomes

$$y^2 = 2px, \tag{2}$$

which is the form of the equation of a parabola which we shall use hereafter.

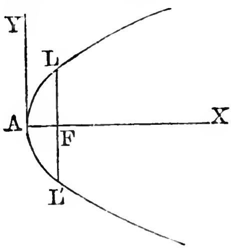
In equation (2), let $x = \frac{1}{2}p$.

Then $y^2 = p^2$,

or $y = \pm\, p$.

Hence $FL = FL' = 2AF$,

and $LL' = 2p \quad = 4AF$.



Def. The double ordinate through the focus is called the principal **parameter** or **latus rectum.**

Cor. *The length of the semi-parameter is p.*

99. *Focal Distance of any Point on the Parabola.*

Let r denote the focal distance FP of any point P (§ 98). Then, by the definition of the curve, we have

$$FP = NP$$
$$= OA + AM,$$

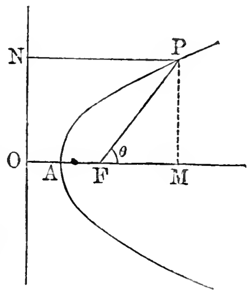
or
$$r = \tfrac{1}{2}p + x, \tag{3}$$

which, being of the first degree, is sometimes called the *linear* equation of the parabola.

100. *Polar Equation of the Parabola.*

Problem. *To find the polar equation of the parabola, the focus being the pole.*

Let $FP = r$, $XFP = \theta$. Then, from the figure, we have



$$FP = PN$$
$$= OF + FM$$
$$= 2AF + FM,$$

or
$$r = p + r \cos\theta;$$

whence
$$r = \frac{p}{1 - \cos\theta} = \frac{p}{2\sin^2\frac{\theta}{2}}. \tag{4}$$

If we count the angle θ from the vertex in the direction AP, we shall have $AFP = \theta$, and therefore (4) becomes

$$r = \frac{p}{1 + \cos\theta} = \frac{p}{2\cos^2\frac{\theta}{2}}, \tag{5}$$

which is the form of the polar equation generally used.

Cor. The polar equation may also be easily deduced from the *linear* equation of the curve. Thus, when the vertex is the origin, the linear equation is

$$r = \tfrac{1}{2}p + x;$$

and transferring the origin to the focus by writing $x + \frac{1}{2}p$ for x, it becomes

$$r = p + x$$
$$= p - r\cos\theta;$$

whence
$$r = \frac{p}{1 + \cos\theta},$$

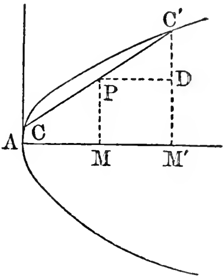
as before.

Diameters of a Parabola.

101. *Def.* The **diameter** of a parabola is the locus of the middle points of any system of parallel chords.

PROBLEM. *To find the equation of any diameter.*

Let x, y be the co-ordinates of P, the middle point of any chord CC'; x', y' the co-ordinates of C'; $r = PC'$, half the length of the chord; θ the inclination of the chord to the axis of the curve. Draw the ordinates PM, $C'M'$ and PD parallel to AM'. Then we have



$$AM' = AM + PD,$$

or
$$x' = x + r\cos\theta, \qquad (a)$$

and
$$C'M' = PM + C'D,$$

or
$$y' = y + r\sin\theta; \qquad (b)$$

and since the point (x', y') is on the curve, we have

$$y'^2 = 2px'. \qquad (c)$$

Substituting the values of x' and y' as given by (a) and (b) in (c), we have

$$(y + r\sin\theta)^2 = 2p(x + r\cos\theta),$$

or
$$r^2\sin^2\theta + 2(y\sin\theta - p\cos\theta)r + y^2 - 2px = 0,$$

from which to determine the two values of r. But since the point (x, y) is the middle of the chord, the values of r are equal in magnitude but of opposite signs; therefore the coefficient of r must vanish, which gives

$$y\sin\theta - p\cos\theta = 0,$$

or
$$y = p\cot\theta$$
$$= \frac{p}{m},$$

where $m = \tan\theta$, the slope of the chord to the axis of X.

Hence the equation of any diameter is

$$y = p\cot\theta. \times \qquad (6)$$

Since the second member of (6) is constant for any system of parallel chords, *every diameter of a parabola is a straight*

× Cf. note, opposite page.

line parallel to the axis of X (§ 40, III.). Because m may have any value whatever, (6) can be made to represent any straight line parallel to the axis of the curve. Hence *every line parallel to the axis bisects a system of parallel chords.*

Cor. To draw a diameter of the curve, bisect any two parallel chords, join the points of bisection and produce the line to meet the curve: it will be a diameter.

Tangents and Normals.

102. PROBLEM. *To find the equation of a tangent to a parabola.*

Let (x', y') and (x'', y'') be the co-ordinates of any two points on the curve. Then the equation of the secant through these points is

$$y - y' = \frac{y'' - y'}{x'' - x'}(x - x'). \qquad (a)$$

But since (x', y') and (x'', y'') are on the curve, we have

$$y'^2 = 2px' \qquad (b)$$

and

$$y''^2 = 2px''. \qquad (c)$$

From (b) and (c) we get

$$y''^2 - y'^2 = 2p(x'' - x');$$

whence

$$\frac{y'' - y'}{x'' - x'} = \frac{2p}{y'' + y'}, \quad *$$

which substituted in (a) gives, for the equation of the secant,

$$y - y' = \frac{2p}{y'' + y'}(x - x').$$

Now when the point (x'', y'') coincides with the point (x', y'), the secant will become a tangent, and then $x'' = x'$ and $y'' = y'$; hence the equation of the tangent at the point (x', y') is

$$y - y' = \frac{p}{y'}(x - x'), \qquad (7)$$

or

$$\begin{aligned} y'y &= px - px' + y'^2 \\ &= px - px' + 2px' \\ &= p(x + x'). \qquad (8) \end{aligned}$$

* If $\frac{2p}{y''+y'}$ = constant, the equation sh…

Cor. Let (x', y') be the co-ordinates of any point on a parabola; the equation of the tangent at that point is

$$y = \frac{p}{y'}(x + x'), \tag{a}$$

and the equation of the diameter passing through the point (x', y') is

$$y' = \frac{p}{m}. \tag{b}$$

Eliminating y' from (a) and (b), we have

$$y = m(x + x')$$

for the equation of the tangent. But m is the slope of the parallel chords to the axis; hence *the tangent at the extremity of any diameter of a parabola is parallel to the chords which are bisected by that diameter.*

The equation of the tangent may also be derived independently of the point of contact in the following manner.

103. PROBLEM. *To find the condition that the line*

$$y = mx + h$$

may be tangent to a given parabola.

The equation of the curve is

$$y^2 = 2px;$$

whence, by eliminating y between these equations, we get

$$(mx + h)^2 = 2px,$$

or
$$m^2x^2 + (2mh - 2p)x + h^2 = 0,$$

which determines the abscissæ x of the two points in which the line intersects the curve. But since the line is to be a tangent, the two values of the abscissæ will be equal. The condition that this equation may have equal roots* is

$$(2mh - 2p)^2 = 4m^2h^2;$$

whence
$$h = \frac{p}{2m},$$

---

* The condition that the roots of the quadratic equation

$$ax^2 + bx + c = 0$$

shall be equal is
$$b^2 - 4ac = 0. \qquad \text{(Chap. I.)}$$

the required condition. Substituting this value of h in the equation of the line, we have

$$y = mx + \frac{p}{2m}, \tag{9}$$

which is the equation of the tangent to a parabola in terms of the slope and semi-parameter.

Conversely, every line whose equation is of this form is a tangent to a parabola.

104. *The Subtangent. Def.* The **subtangent** is the projection of the tangent upon the axis of the parabola.

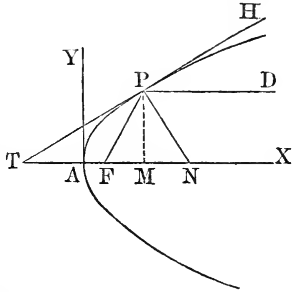
To find where the tangent meets the axis of X, make $y = 0$, in (8), and we get

$$0 = p(x + x'),$$

or $$x = -x';$$

that is, $$AT = AM;$$

or, *the subtangent is bisected at the vertex.*



This property enables us to draw a tangent at any point on a parabola. Thus, let P be any point on the curve; draw the ordinate PM, and produce MA to T, making AT equal to AM; join TP. Then TP is the tangent required.

105. *The Normal. Def.* The **normal** to a curve at any point is the perpendicular to the tangent at that point.

PROBLEM. *To find the equation of the normal to a parabola.*

The equation of the tangent at any point (x', y') has been shown to be

$$y = \frac{p}{y'}(x + x'); \tag{a}$$

and let $$y - y' = m(x - x') \tag{b}$$

be the equation of a line through (x', y') and normal to the curve at that point.

Now in order that the lines represented by (a) and (b)

may be perpendicular to each other, we must have the condition

$$\frac{pm}{y'} + 1 = 0, \qquad (\S\ 47)$$

or

$$m = -\frac{y'}{p};$$

therefore (*b*) becomes

$$y - y' = -\frac{y'}{p}(x - x'), \qquad (10)$$

which is the equation of the normal at the point (x', y').

106. *The Subnormal. Def.* The **subnormal** is the projection of the normal upon the axis of the parabola.

To find where the normal intersects the axis of X, make $y = 0$ in (10). Then we have

$$\begin{aligned} p &= x - x' \\ &= AN - AM \\ &= MN; \end{aligned}$$

that is, *the subnormal MN is constant and equal to half the parameter or latus rectum.*

107. THEOREM. *A tangent to a parabola is equally inclined to the axis of the curve and the focal line from the point of tangency.*

Proof. From (§ 104) we have

$$\begin{aligned} FT &= AT + AF \\ &= AM + AF \\ &= FP; \end{aligned}$$

therefore the angle PTF is equal to the angle FPT.

Cor. Let PD be drawn parallel to the axis AX; then PD is a diameter of the curve (§101), and the angle HPD is equal to the angle TPF. Since the normal PN is perpendicular to the tangent, the angle DPN is equal to the angle NPF.

REMARK. The properties just proved find an application in the use of parabolic reflectors intended to bring rays of light to a focus, as in the reflecting telescope. Since the curve and the tangent have the same direction at the point of tangency, rays of light are reflected by the

curve as they would be by the tangent at that point; and because the angle of incidence is equal to the angle of reflection, it follows that if rays of light parallel to the axis of the curve fall upon a parabolic reflector, they will all be reflected to the focus. Conversely, if a luminous body be placed in the focus of a parabolic reflector, all the rays proceeding therefrom will be parallel after reflection.

108. PROBLEM. *To find the locus of the foot of the perpendicular from the focus upon a variable tangent.*

Let x', y' be the co-ordinates of any point P on the curve.

The equation of the tangent at P is

$$y = \frac{p}{y'}(x + x'). \tag{a}$$

The equation of the line through the focus whose co-ordinates are $(\frac{1}{2}p,\ 0)$, and perpendicular to (a), is

$$y = -\frac{y'}{p}\left(x - \frac{p}{2}\right). \tag{b}$$

And since the point $(x',\ y')$ is on the curve

$$y'^2 = 2px', \tag{c}$$

we have now to eliminate x' and y' from (a), (b) and (c).

From (c),

$$x' = \frac{y'^2}{2p},$$

which substituted in (a) gives

$$y = \frac{p}{y'}\left(x + \frac{y'^2}{2p}\right). \tag{d}$$

From (b) we have

$$y' = -\frac{py}{x - \frac{1}{2}p},$$

which substituted in (d) gives, after obvious reductions,

$$\{y^2 + (x - \tfrac{1}{2}p)^2\}x = 0.$$

Therefore we must have either

$$y^2 + (x - \tfrac{1}{2}p)^2 = 0 \quad \text{or} \quad x = 0.$$

The former gives $y = 0$ and $x = \frac{1}{2}p$, the focus, which however is not the locus of the intersection of (a) and (b);

for although these values of x and y satisfy (b), they do not satisfy (a). We conclude, therefore, that the latter, namely,

$$x = 0, \tag{11}$$

is the equation of the required locus, which is the tangent at the origin or the axis of Y.

109. PROBLEM. *To find the locus of the point of intersection of two tangents to a parabola which are perpendicular to each other.*

Let the equation of one of the tangents be

$$y = mx + \frac{p}{2m}. \qquad (\S\,103) \qquad (a)$$

Then the equation of the other, perpendicular to (a), is

$$y = -\frac{x}{m} - \tfrac{1}{2}pm,$$

or

$$my = -x - \tfrac{1}{2}pm^2. \qquad (b)$$

Multiplying (a) by m and subtracting (b) gives

$$0 = 2(1 + m^2)x + (1 + m^2)p,$$

or

$$x = -\tfrac{1}{2}p, \tag{12}$$

the equation of the required locus, which is the directrix.

110. PROBLEM. *To find the length of the perpendicular from the focus upon the tangent at any point.*

Let P denote the length of the perpendicular. The equation of the tangent at the point (x', y') is

$$y'y - p(x + x') = 0. \qquad (\S\,102)$$

The perpendicular P from the focus, whose co-ordinates are $(\frac{1}{2}p, 0)$, is (§ 41)

$$P = \frac{p(p + 2x')}{2\sqrt{y'^2 + p^2}} = \frac{p(2x' + p)}{2\sqrt{2px' + p^2}}$$

$$= \tfrac{1}{2}\sqrt{p(p + 2x')}$$

$$= \tfrac{1}{2}\sqrt{2pr}, \qquad (\S\,99) \qquad (13)$$

where r is the focal distance of the point of tangency.

111. PROBLEM. *To find the co-ordinates of the point of contact of a tangent drawn from a given point to a parabola.*

Let (x', y') be the co-ordinates of the required point of contact, and (h, k) the co-ordinates of the given fixed point. The equation of the tangent at (x', y') is

$$y'y = p(x + x');$$

and since the tangent passes through the point (h, k), we also have

$$ky' = p(x' + h). \tag{a}$$

Because the point (x', y') is on the curve, we have

$$y'^2 = 2px'. \tag{b}$$

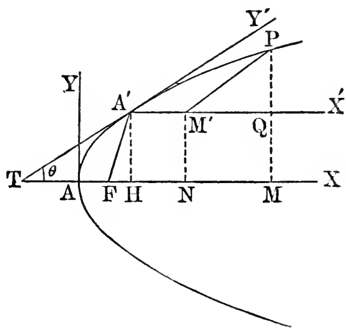
Solving (a) and (b) for x' and y', we have

$$x' = k^2 - 2ph \pm k\sqrt{k^2 - 2ph};$$
$$y' = k \pm \sqrt{k^2 - 2ph}.$$

These equations show that from any fixed point *two* tangents can be drawn to a parabola, and that the points of contact (x', y') will be *real, coincident* or *imaginary* according as $k^2 - 2ph > 0$, $= 0$, or < 0; that is, according as the point (h, k) is *without, on* or *within* the curve.

112. PROBLEM. *To find the equation of the parabola referred to any diameter and the tangent at its vertex, as axes.*

Let A' be any point on a parabola; take this point as origin and draw through it the diameter $A'X'$ for the new axis of X, and the tangent $TA'Y'$ for the new axis of Y.



Let $Y'A'X' = A'TX = \theta$, and h and k the co-ordinates of A' referred to the original axes AX, AY.

Let (x, y) be the co-ordinates of any point P referred to the *original* axes, and (x', y') the co-ordinates of the same point referred to the *new* axes; draw the ordinates PM, PM', and draw $M'N$ and $A'H$ paral-

lel to AY, and let Q denote the intersection of the diameter $A'X'$ and the ordinate PM. Then, from the figure, we have

$$\begin{aligned} x = AM &= AH + A'M' + M'Q \\ &= h + x' + PM' \cos PM'Q \\ &= h + x' + y' \cos \theta, \end{aligned} \tag{a}$$

and

$$\begin{aligned} y = PM &= A'H + PQ \\ &= k + PM' \sin PM'Q \\ &= k + y' \sin \theta. \end{aligned} \tag{b}$$

But since the point (x, y) is on the curve,

$$y^2 = 2px. \tag{c}$$

Substituting the values of x and y as given by (a) and (b) in (c), we have

$$(k + y' \sin \theta)^2 = 2p(h + x' + y' \cos \theta);$$

whence

$$y'^2 \sin^2 \theta + 2y'(k \sin \theta - p \cos \theta) + k^2 - 2ph = 2px'.$$

But, by (6) of § 101,

$$k = p \cot \theta;$$

and since the point (h, k) is on the curve

$$k^2 = 2ph,$$

therefore we have

$$y'^2 \sin^2 \theta = 2px',$$

or

$$y'^2 = \frac{2p}{\sin^2 \theta} x', \tag{14}$$

which is the equation of the curve referred to the new axes.

Cor. This equation may also be expressed in terms of $A'F$, the focal distance of the point A'. Thus, by (3) of §99,

$$A'F = \tfrac{1}{2}p + h,$$

and

$$k^2 = 2ph, \qquad \text{or} \qquad h = \frac{k^2}{2p}.$$

Therefore

$$\begin{aligned} A'F &= \tfrac{1}{2}p + \frac{k^2}{2p} \\ &= \tfrac{1}{2}(p + p \cot^2 \theta) \qquad (\text{since } k = p \cot \theta) \\ &= \tfrac{1}{2}p(1 + \cot^2 \theta) \\ &= \frac{p}{2 \sin^2 \theta}. \end{aligned}$$

Therefore, denoting $A'F$ by $\frac{1}{2}p'$, equation (14) may be written

$$y'^2 = 2p'x',$$

or, suppressing the accents on the variables,

$$y^2 = 2p'x. \tag{15}$$

Cor. From the identity of form in the equations

$$y^2 = 2px \qquad \text{and} \qquad y^2 = 2p'x,$$

we may at once infer that the equation of the tangent referred to any diameter is

$$y'y = p'(x + x'). \tag{16}$$

If in this equation we put $y = 0$, we get

$$x = -x';$$

or, the intercept on the axis of X is equal to the abscissa of the point of contact, and therefore the subtangent to any diameter of a parabola is bisected by the vertex.

Poles and Polars.

113. *Def.* **A chord of contact** is the chord joining the points of contact of two tangents.

PROBLEM. *To find the equation of the chord of contact of two tangents from an external point.*

Let (x_1, y_1) be the co-ordinates of the external point, (x', y') the co-ordinates of the point where one of the tangents through (x_1, y_1) meets the curve, and (x'', y'') the co-ordinates of the point where the other tangent meets the curve.

The equation of the tangent at (x', y') is

$$y'y = p(x' + x); \tag{a}$$

and since this passes through (x_1, y_1), we have

$$y_1y' = p(x' + x_1), \tag{b}$$

and, for the same reason,

$$y_1y'' = p(x'' + x_1). \tag{c}$$

Hence the equation of the chord of contact is

$$y_1y = p(x + x_1), \tag{17}$$

for this is the equation of a straight line, and is satisfied by

$$x = x', \qquad y = y' \qquad \text{and} \qquad x = x'', \qquad y = y'',$$

as we see from (*b*) and (*c*). Therefore (17) *is the equation of the chord of contact of the tangents through the point* (x_1, y_1).

114. *Locus of the Point of Intersection of Two Tangents.* Let (x_2, y_2) be the co-ordinates of any fixed point through which a chord of contact of two intersecting tangents is drawn, and (x_1, y_1) the co-ordinates of the point of intersection of the tangents. Then the equation of the chord of contact is, by the last section,

$$y_1 y = p(x_1 + x);$$

but since (x_2, y_2) is a point on the chord, we must also have the condition

$$y_1 y_2 = p(x_1 + x_2),$$

which the co-ordinates of the point of intersection must always satisfy, however the chord of contact may change its position as it revolves about the fixed point (x_2, y_2). Therefore the equation of the required locus is

$$y_2 y = p(x + x_2), \tag{18}$$

which is that of a straight line. Hence we have the theorem:

If through any fixed point we draw chords to a parabola;

and if through the ends of each chord we draw a pair of tangents,

then the point of meeting of every pair of tangents will lie on a certain straight line.

Def. Such straight line is called the **polar** of the point through which the chords pass.

It follows from this theorem that if (x_2, y_2) be any fixed point, the equation of the *polar* of that point is

$$y_2 y = p(x + x_2) \tag{19}$$

when referred to the *axis*, or

$$y_2 y = p'(x + x_2) \tag{20}$$

if referred to a *diameter* and a *tangent* at its vertex as axes.

Direction of the Polar.

Making $y_2 = 0$ in (20), we have

$$x = -x_2, \tag{21}$$

which is the equation of a line parallel to the axis of Y. Hence

The polar of any point is parallel to the tangent at the end of the diameter passing through that point, and is situated at a distance from the vertex of the diameter equal, but in an opposite direction, to the distance of the point.

115. *Polar of the Focus.*

Put $(\frac{1}{2}p, 0)$ for x_2, y_2 in the equation of the polar, and we get

$$x = -\tfrac{1}{2}p, \tag{22}$$

which is the equation of the directrix. Therefore

The polar of the focus of a parabola is the directrix.

EXERCISES.

1. Find the points of intersection of the line $y = 3x - 6$ with the parabola $y^2 = 9x$. *Ans.* (4, 6) and (1, − 3).

2. Find the equation of a line through the focus of the parabola $y^2 = 12x$ and making an angle of 30° with the axis of x.

$$\textit{Ans.}\quad y = \frac{x}{\sqrt{3}} - \sqrt{3}.$$

3. Find the equation of the line through the vertex and the extremity of the latus rectum. *Ans.* $y = \pm 2x$.

4. Find the equation of the circle which passes through the vertex of a parabola and the extremities of the latus rectum. *Ans.* $x^2 + y^2 = \frac{5}{2}px$.

5. Find the equation of the tangent at the extremity of the latus rectum, and the angle between this tangent and the line drawn to the vertex from the same extremity of the latus rectum. *Ans.* $y = x + \frac{1}{2}p$; $\tan^{(-1)}\frac{1}{3}$.

6. Determine the equations of the normals at the extremities of the latus rectum, the co-ordinates of the points in which these normals again intersect the curve, and the length of the chords formed by the normals.

$$\textit{Ans.}\quad y \pm x = \frac{3p}{2};\quad \left(\frac{9p}{2}, \mp 3p\right);\quad 4p\sqrt{2}.$$

7. Show that if the focus of a parabola is the origin, and the axis of the curve the axis of X, the equation of the parabola is $y^2 = p(2x + p)$, and the equation of the tangent at the point (x', y') is

$$y'y = p(x + x' + p).$$

8. With the same origin and axes as in the last example show that the equations of the tangents and normals at the extremity of the latus rectum are

$$x \mp y + p = 0;$$
$$x \pm y - p = 0.$$

9. Prove that the circle described on any focal chord as diameter will touch the directrix.

10. A tangent is drawn to a parabola at the point (x', y'). Find the length of the perpendicular drawn from the foot of the directrix on this tangent.

Ans. $\dfrac{y'^2 - p^2}{2\sqrt{y'^2 + p^2}}$.

11. Pairs of tangents are drawn to a parabola at points whose abscissæ are in a constant ratio. Show that the locus of the intersection of the tangents is a parabola.

12. Find the polar equation of the parabola when the vertex is the pole, and the axis of the curve the initial line.

Ans. $r = 2p \cot\theta \operatorname{cosec}\theta$.

13. If r and r' be the lengths of two radii vectores drawn at right angles to each other from the vertex of a parabola, show that

$$(rr')^{\frac{4}{3}} = 4p^2(r^{\frac{2}{3}} + r'^{\frac{2}{3}}).$$

14. Find the equation of the parabola referred to the tangents at the extremities of the latus rectum as axes.

Ans. $(x - y)^2 - 2\sqrt{2}p(x + y) + 2p^2 = 0$.

15. If tangents be drawn to a parabola at the extremities of any focal chord, show that they will intersect at right angles on the directrix, and that the line from their point of intersection to the focus is perpendicular to the focal chord.

$BC = p \cot \theta$ Coor of D $(-\frac{1}{2}p, p\cot\theta)$

$FP.$ $y = \tan\theta(x - \frac{1}{2}p)$

16. From an external point (x', y') two tangents are drawn to a parabola. Show that the length of the chord of contact is

$$\frac{2(y'^2 + p^2)^{\frac{1}{2}}(y'^2 - 2px')^{\frac{1}{2}}}{p},$$

and that the area of the triangle formed by the chord and tangents is

$$\frac{(y'^2 - 2px')^{\frac{3}{2}}}{p}.$$

17. If m, m' be the slopes to the axis of the parabola of the two tangents in the last example, show that

$$m + m' = \frac{y'}{x'} \qquad \text{and} \qquad mm' = \frac{p}{2x'}.$$

18. If (x', y') and (x'', y'') be any two points on a parabola, show that the tangent of the angle contained by the tangents touching at these points is

$$\frac{p(y'' - y')}{p^2 + y''y'}$$

19. In what ratio does the focus of a parabola divide that segment of the axis cut out by a tangent and normal drawn at the same point of the parabola?

20. A triangle is formed by three tangents to a parabola. Show that the circle which circumscribes this triangle passes through the focus.

21. Show that the parameter of any diameter is equal to four times the focal distance of its vertex, or equal to the focal double ordinate of that diameter.

NOTE. The parameter of any diameter is the focal chord bisected by that diameter, called $2p'$ in § 112.

22. If TP and TQ are tangents to a parabola at the points P and Q, then if F be the focus, show that

$$FP \,.\, FQ = FT^2.$$

23. Show that the area of the triangle in Prob. 20 is half that of the triangle formed by joining the points of contact of the three tangents.

24. Given the outline of a parabola, show how to find the focus and the axis.

25. The base of a triangle is $2a$, and the sum of the tangents of the base-angles is m. Show that the locus of the vertex is a parabola whose semi-parameter is $\frac{a}{m}$.

26. Prove that $y = x \tan \theta + p \operatorname{cosec} 2\theta$ is a tangent to a parabola whose latus rectum is p, the origin being at the focus, and the axis of the curve the axis of X.

27. Tangents are drawn from any two points P, Q to a parabola. Show that the co-ordinates of T, the intersection of the tangents, are

$$\frac{1}{4}p\,\frac{\cos(\theta_1 + \theta_2)}{\sin\theta_1 \sin\theta_2}, \qquad \frac{1}{4}p\,\frac{\sin(\theta_1 + \theta_2)}{\sin\theta_1 \sin\theta_2},$$

where $\tan\theta_1$ and $\tan\theta_2$ are the slopes of the tangents to the axis of X.

28. If all the ordinates of a parabola are increased in the same ratio, show that the new curve will be a parabola, and express its parameter in terms of the ratio of increase.

29. At what point of a parabola is the normal double the subtangent; and what angle does that normal form with the axis of the parabola?

30. Find a point upon a parabola such that the rectangle contained by the tangent and normal shall be twice the square of the ordinate; and show the relation of such point to the focus.

31. Find that point on a parabola for which the normal is equal to the difference between the subtangent and the subnormal.

32. Having given the parabola $y^2 = 6x$, find the equation of that chord which is bisected by the point (4, 3).

33. Find the equation of that chord of a parabola which is drawn from the vertex and bisected by the diameter $y = q$.

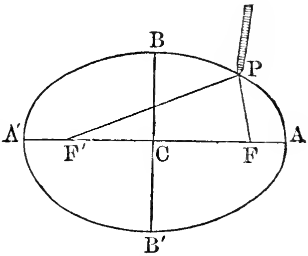
CHAPTER VI.

THE ELLIPSE.

Equations and Fundamental Properties.

116. *Def.* An **ellipse** is the locus of a point the sum of whose distances from two fixed points is constant.

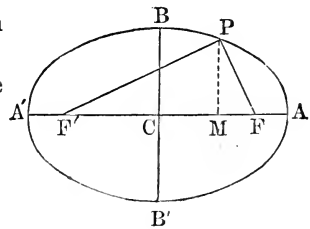
The two fixed points are called **foci** of the ellipse. Thus, if the point P move in such a way that $PF + PF'$ is constant, it will describe an ellipse.



The curve may be described mechanically as follows: Take any two fixed points F and F', and attach to them the extremities of a thread whose length is greater than the distance FF'. Place a pencil-point P against the thread, and slide it so as to keep the thread constantly stretched: the point P will describe an ellipse, for in every position we shall have $PF + PF' =$ the constant length of the thread.

The line AA' drawn through the foci and terminated by the curve is called the **transverse** or **major axis,** and BB' bisecting AA' at right angles is called the **conjugate** or **minor axis.** The two are called **principal axes.**

The semi-axes CA and CB are represented by the symbols a and b respectively.



The point C midway between the foci is called the **centre.**

From the manner in which the curve is generated, we see that

$$AF = A'F'$$

and

$$PF + PF' = AA'.$$

117. PROBLEM. *To find the equation of the ellipse.*

Solution. Let C, the intersection of AA' and BB', be the origin; CA the axis of X, and CB the axis of Y; put $CA = CA' = a$, $CF = CF' = c$, and x, y the co-ordinates of any point P on the locus. Then we shall have

$$PF = \sqrt{PM^2 + MF^2} = \sqrt{y^2 + (c - x)^2};$$
$$PF' = \sqrt{PM^2 + MF'^2} = \sqrt{y^2 + (c + x)^2}.$$

Therefore, by definition,

$$\sqrt{y^2 + (c - x)^2} + \sqrt{y^2 + (c + x)^2} = 2a.$$

Clearing this equation of surds, it reduces to

$$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2).$$

But, by definition,

$$a^2 - c^2 = BF^2 - CF^2 = BC^2 = b^2;$$

therefore we have, by substituting in the above,

$$b^2x^2 + a^2y^2 = a^2b^2;$$

or, dividing by a^2b^2, we have

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad (1)$$

which is the simplest form of the equation of the ellipse. It is called *the equation of the ellipse referred to its centre and axes,* because the centre is the origin and the axes are the axes of co-ordinates.

Def. The distance $CF = CF' = c$ between the centre and either focus is the **linear eccentricity** of the ellipse.

The ratio $\frac{c}{a}$ of the linear eccentricity to the semi-major axis is called the **eccentricity** of the ellipse.

By the common notation,

$$e \equiv \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a} \qquad (2)$$

is the expression for the eccentricity in terms of the semi-axes.

Cor. If we transfer the origin to A', whose co-ordinates are $(-a, 0)$, the equation (1) becomes, by writing $(x - a)$ for x,

$$\frac{(x-a)^2}{a^2} + \frac{y^2}{b^2} = 1,$$

or

$$y^2 = \frac{b^2}{a^2}(2ax - x^2),$$

a form of the equation of the ellipse which is sometimes useful.

EXERCISES.

1. Find the eccentricity and semi-axes of the ellipse

$$16x^2 + 25y^2 = 400.$$

REMARK. Reduce the second member to unity by dividing by 400, and compare with (1).

2. What are the semi-axes and the equation of the ellipse when the distance between the foci is 2 and the sum of the distances from each point of the curve to the foci is 4?

3. Determine the eccentricity and semi-axes of the ellipses having the following equations:

(*a*) $x^2 + 2y^2 = 6$; (*b*) $3x^2 + 4y^2 = 9$; (*c*) $4x^2 + 9y^2 = 16$;
(*d*) $mx^2 + ny^2 = p$; (*e*) $\frac{2}{3}x^2 + \frac{4}{5}y^2 = \frac{1}{2}$; (*f*) $ax^2 + by^2 = 1$.

4. Using the preceding notation, prove the following propositions:

I. The distance of either focus from the centre is ae.

II. The distance of either focus from the nearest end of the major axis is $a(1 - e)$.

III. The distance of either focus from the farther end of the major axis is $a(1 + e)$.

IV. The distance from either end of the major axis to either end of the minor axis is $a\sqrt{2 - e^2}$.

V. If we define an angle φ by the equation

$$\sin \varphi = e,$$

we shall have for the semi-minor axis

$$b = a \cos \varphi.$$

5. Find the points in which the circle $x^2 + y^2 = 4$ intersects the ellipse $x^2 + 2y^2 = 6$.

6. Write the equation of that ellipse whose minor axis is 10 and the distance between whose foci is 12.

118. If we solve equation (1) with respect to y, we find

$$y = \pm \frac{b}{a} \sqrt{a^2 - x^2}.$$

This equation shows that for every value of x there will be two values of y, equal but with opposite signs. *Hence the curve is symmetrical with respect to the major axis.*

By solving with respect to x we show in like manner that *the curve is symmetrical with respect to the minor axis.*

Def. A **chord** of an ellipse is any straight line terminated by two points of the ellipse.

A **diameter** of an ellipse is any chord through its centre.

Cor. The major and minor axes are diameters.

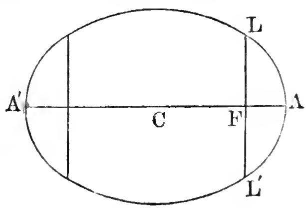
Def. The **parameter** or **latus rectum** is a chord through the focus and perpendicular to the major axis.

119. THEOREM I. *The parameter of an ellipse is a third proportional to the major and minor axes.*

Proof. The semi-parameter is, by definition, the value of the ordinate y when $x = ae$. From equation (1), we have

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2).$$

If we put p for the semi-parameter, we find, by substitution in this equation,



$$p^2 = \frac{b^2}{a^2}(a^2 - a^2e^2) = \frac{b^4}{a^2}.$$

Hence $p = \dfrac{b^2}{a}$, or $ap = b^2$,

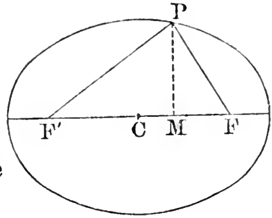
or $a : b = b : p$.

Cor. The length of the semi-parameter FL is

$$p = a(1 - e^2). \qquad (3)$$

120. *Focal Radii, or Radii Vectores.*

Def. The **focal radii** of an ellipse are the lines drawn from any point on the curve to the foci.



PROBLEM. *To express the lengths of the focal radii in terms of the abscissa of the point from which they are drawn.*

Let r and r' be the focal radii of the point P, whose co-ordinates are (x, y).

Because $FC = CF' = ae$,

we have
$$\begin{aligned} r^2 &= FM^2 + PM^2 \\ &= (x - ae)^2 + y^2 \\ &= (x - ae)^2 + \frac{b^2}{a^2}(a^2 - x^2) \\ &= x^2 - 2aex + a^2e^2 + (1 - e^2)(a^2 - x^2) \\ &= a^2 - 2aex + e^2x^2. \end{aligned}$$

Therefore $r = a - ex$. (4)

In the same way we find, for the other focal radius,

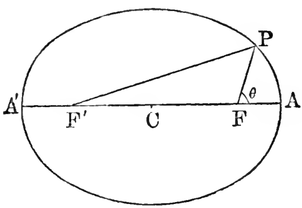
$$r' = a + ex. \qquad (5)$$

These expressions are of remarkable simplicity, and, being of one dimension in x, either of them is called the *linear* equation of the ellipse.

We observe that their sum is $2a$, as it should be.

Cor. Equations (4) and (5) show that *if a point move on the circumference of an ellipse in such a way that its abscissa increases uniformly, one focal radius will increase and the other will decrease uniformly.*

In other words, *if the abscissæ of several points are in arithmetical progression, their focal radii will also be in arithmetical progression.*



121. *Polar Equation of the Ellipse, the right-hand focus being the pole.*

Let r and θ be the polar co-ordinates of any point P on an ellipse; that is, $r = FP$ and

$\theta =$ the angle AFP. Join PF'. Then, from the triangle FPF', we have

$$PF'^2 = PF^2 + FF'^2 - 2PF.\ FF'\ .\cos PFF'.$$

But $\quad FF' = 2ae \quad$ and $\quad \cos PFF' = -\cos AFP$;

therefore $\quad PF' = \sqrt{r^2 + 4a^2e^2 + 4aer \cos \theta}$,

and by the fundamental property of the ellipse we have

$$PF + PF' = AA',$$

or $$r + \sqrt{r^2 + 4a^2e^2 + 4aer \cos \theta} = 2a;$$

whence we easily find

$$r = \frac{a(1 - e^2)}{1 + e \cos \theta}, \tag{6}$$

which is the required equation.

The polar equation may also be easily obtained from the *linear* equation of the ellipse; thus, from (4), we have

$$r = a - ex,$$

the origin being at the centre.

Transferring the origin to the right-hand focus, whose co-ordinates are $(ae, 0)$, it becomes

$$r = a(1 - e^2) - ex,$$

which in polar co-ordinates becomes

$$r = a(1 - e^2) - er \cos \theta;$$

whence $$r = \frac{a(1 - e^2)}{1 + e \cos \theta},$$

as before.

If the left-hand focus be taken as the pole, the student may easily show that the polar equation is

$$r = \frac{a(1 - e^2)}{1 - e \cos \theta}.$$

Cor. If $\theta = 0$, we have $r = \dfrac{a(1 - e^2)}{1 + e} = a(1 - e)$, which is the value of AF.

When $\theta = 180°$, we get $r = a(1 + e)$, which is the value of $A'F$.

When $\theta = 90°$, $r = a(1 - e^2)$, the semi-parameter.

These results agree with those of §§ 117, 119.

EXERCISES.

1. If the semi-minor axis of an ellipse is b, and the eccentricity $\sin \varphi$, express its semi-major axis and semi-parameter in terms of b and φ. *Ans.* $a = b \sec \varphi$; $p = b \cos \varphi$.

2. The distance from the focus to the nearer end of the major axis is 2, and the semi-parameter is 3. Find the major and minor axes and the eccentricity.

3. Express the ratio of the parameter to the distance between the focus and either end of the major axis.

4. The major axis is divided by the focus into two segments. Show that the rectangle contained by these segments is equal to the rectangle contained by the semi-major axis and the semi-parameter, and also equal to the square of the semi-minor axis.

5. Write the equation of an ellipse in terms of its semi-minor axis b, and its semi-parameter p.

Ans. $p^2x^2 + b^2y^2 = b^4$.

6. Find the points in which the several straight lines

$$y = 2x, \qquad y = 2x + 1, \qquad y = 2x + 2,$$

intersect the ellipse $x^2 + 2y^2 = 6$, and the lengths of the three chords which the ellipse cuts out from the lines.

7. Find the equation of the ellipse when the right-hand focus is the origin, the axes being the major axis and the latus rectum.

$$\textit{Ans.}\ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{2ex}{a} = \frac{b^2}{a^2}.$$

8. The sum of the principal axes of an ellipse is 108, and the linear eccentricity 36. Find the equation of the ellipse, and the eccentricity.

$$\textit{Ans.}\ \frac{x^2}{39^2} + \frac{y^2}{15^2} = 1; \quad e = \frac{12}{13}.$$

Diameters of an Ellipse.

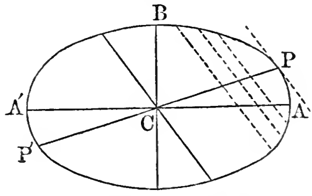
122. THEOREM II. *Every diameter of an ellipse is bisected by the centre.*

Proof. Let $y = mx$ be the equation of any line through the centre. Eliminating y between this equation and that of the ellipse, we have

$$\frac{x^2}{a^2} + \frac{m^2x^2}{b^2} = 1$$

from which to determine the abscissæ of the points in which the line intersects the ellipse. Since this equation contains terms in x^2 but none in x, it will reduce to a pure quadratic, of which the two roots are equal but with opposite signs. From these roots we shall get, by substituting in the equation $y = mx$, two equal values of y with opposite signs. Hence the points of intersection are at equal distances on each side of the origin.

123. THEOREM III. *The locus of the centres of parallel chords of an ellipse is a diameter.*



Proof. Let $y = mx + h$ (a) be the equation of a chord; the slope m being the same for all the chords, while h varies from one chord to another.

We first find the points of intersection of the chord with the ellipse in the usual way.

Eliminating y between (a) and the equation of the ellipse, we find the abscissæ of the points of intersection to be determined by the quadratic equation

$$\frac{x^2}{a^2} + \frac{(mx + h)^2}{b^2} = 1,$$

which being reduced to the general form becomes

$$x^2 + \frac{2a^2mh}{a^2m^2 + b^2}x + \frac{a^2(h^2 - b^2)}{a^2m^2 + b^2} = 0.$$

Now we need not actually solve this equation to obtain

the result we want, namely, the abscissa of the middle point of the chord. We know that if we put, for brevity,

$$p = \frac{2a^2mh}{a^2m^2 + b^2},$$
$$q = \frac{a^2(h^2 - b^2)}{a^2m^2 + b^2},$$

and call the roots x_1 and x_2, we shall have

$$x_1 = \tfrac{1}{2}(-p - \sqrt{p^2 - 4q}),$$
$$x_2 = \tfrac{1}{2}(-p + \sqrt{p^2 - 4q}),$$

which give the abscissæ of the two points in which the chord intersects the ellipse. The corresponding values of y, from (a), are

$$y_1 = mx_1 + h;$$
$$y_2 = mx_2 + h.$$

By (§ 23), the co-ordinates of the middle point of the chord are the half-sums of the co-ordinates of the extremities. If, then, we put x', y' for the co-ordinates of the middle point of the chord, we have

$$\begin{aligned} x' &= -\tfrac{1}{2}p = -\frac{a^2mh}{a^2m^2 + b^2}, \\ y' &= \tfrac{1}{2}m(x_1 + x_2) + h \\ &= mx' + h \\ &= -\frac{a^2m^2h}{a^2m^2 + b^2} + h \\ &= \frac{b^2h}{a^2m^2 + b^2}. \end{aligned} \qquad (b)$$

The problem now is, What relation exists between x' and y' when we suppose h to vary and all the other quantities which enter the second member of (b) to remain constant? We obtain this relation by eliminating h between the two equations, which is done by multiplying the first by b^2 and the second by a^2m and adding the products. Thus we find

$$b^2x' + a^2my' = 0. \times \qquad (7)$$

This is a relation between the co-ordinates of the middle points of the parallel chords which is true for all values of h,

× More simply obtained from (b) page 149:

$b^2 \frac{x'+x''}{\ldots}$ b^2x

that is, for all such chords; it is therefore the equation of the required locus, and, from its form, is a straight line through the origin and therefore through the centre of the ellipse.

124. *Conjugate Diameters.* If we omit the accents in (7), we may write it in the form

$$y = -\frac{b^2}{a^2m}x.$$

By assigning different values to m, or, which is the same thing, by giving different directions to the parallel chords, the slope $-\frac{b^2}{a^2m}$ may take all possible values, and therefore (7) may represent any line passing through the centre and bisecting a system of parallel chords.

If m' be the slope of the diameter which bisects all the chords whose slope is m, we have

$$y = m'x,$$

the equation of the diameter;

but, by (7), $$y = -\frac{b^2}{a^2m}x$$

is also the equation of the diameter.

Therefore $$m' = -\frac{b^2}{a^2m},$$

or $$mm' = -\frac{b^2}{a^2}. \tag{8}$$

Theorem IV. *If one diameter bisects chords parallel to a second diameter, the second diameter will bisect all chords parallel to the first.*

Proof. If m and m' be the respective slopes of the two diameters, we shall have

$$mm' = -\frac{b^2}{a^2},$$

since the first bisects all chords parallel to the second; but this is also the only condition which must hold in order that the second may bisect all chords parallel to the first.

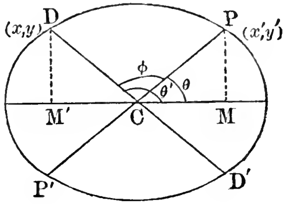
Def. Two diameters each of which bisects all chords parallel to the other are called **conjugate diameters.**

Cor. As the chords of a set become indefinitely short near the terminus of the bisecting diameter, they coincide in *direction* with the tangent at the terminus. Hence:

THEOREM V. *The tangent to an ellipse at the end of a diameter is parallel to the conjugate diameter.*

125. PROBLEM. *Given the co-ordinates of the extremity of one diameter, to find those of either extremity of the conjugate diameter.*

Solution. Let PCP' and DCD' be any pair of conjugate diameters, and (x', y') the given co-ordinates of P. Then the equation of CP is



$$y = \frac{y'}{x'}x, \qquad \left(\text{since } m = \frac{y'}{x'}\right)$$

and the equation of DD' is

$$y = -\frac{b^2}{a^2m}x \qquad (a)$$

or

$$y = -\frac{b^2x'}{a^2y'}x; \qquad (b)$$

and the equation of the ellipse,

$$a^2y^2 + b^2x^2 = a^2b^2. \qquad (c)$$

Substituting from (b) in (c), we have

$$(b^2x'^2 + a^2y'^2)x^2 = a^4y'^2;$$

but since (x', y') is on the ellipse, we have

$$b^2x'^2 + a^2y'^2 = a^2b^2;$$

therefore

$$a^2b^2x^2 = a^4y'^2,$$

or

$$x = \pm\frac{a}{b}y'.$$

Substituting this value of x in (a), we get

$$y = \mp\frac{b}{a}x'.$$

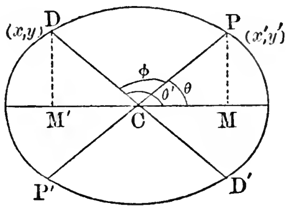
126. THEOREM VI. *The sum of the squares of two conjugate semi-diameters is constant and equal to the sum of the squares of the semi-axes.*

Proof. Let (x', y') be the co-ordinates of P (last figure), and denote the semi-conjugate axes CP, CD by a' and b' respectively. Then we shall have

$$CP^2 + CD^2 = x'^2 + y'^2 + \frac{a^2}{b^2}y'^2 + \frac{b^2}{a^2}x'^2$$
$$= \frac{a^2y'^2 + b^2x'^2}{b^2} + \frac{a^2y'^2 + b^2x'^2}{a^2}$$
$$= \frac{a^2b^2}{b^2} + \frac{a^2b^2}{a^2},$$

or $$a'^2 + b'^2 = a^2 + b^2. \tag{9}$$

127. PROBLEM. *To find the angle between two conjugate axes.*



Let θ and θ' be the angles which the semi-conjugate axes make with the major axis, and φ the angle between the conjugate axes. Then

$$\varphi = \theta' - \theta$$

and

$$\sin\varphi = \sin\theta'\cos\theta - \sin\theta\cos\theta'. \tag{a}$$

Denote the semi-conjugate axes by a' and b', and the co-ordinates of P by x', y'. Then (§ 125) the co-ordinates of D are

$$-\frac{a}{b}y', \qquad +\frac{b}{a}x'.$$

Hence $$\sin\theta = \frac{y'}{a'}; \qquad \cos\theta = \frac{x'}{a'};$$
$$\sin\theta' = \frac{bx'}{ab'}; \qquad \cos\theta' = -\frac{ay'}{bb'}.$$

Substituting in (a), we have

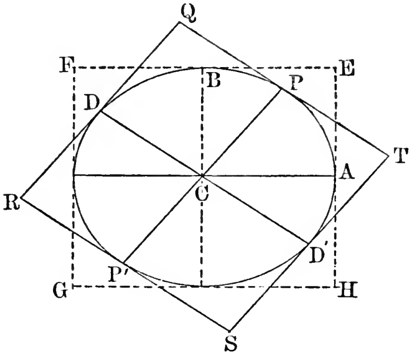
$$\sin\varphi = \frac{bx'^2}{aa'b'} + \frac{ay'^2}{a'bb'} = \frac{b^2x'^2 + a^2y'^2}{aba'b'}.$$

But since (y', y') is on the curve,

$$b^2x'^2 + a^2y'^2 = a^2b^2.$$

Therefore $$\sin \varphi = \frac{a^2b^2}{aba'b'} = \frac{ab}{a'b'}. \qquad (10)$$

128. Theorem VII. *The area of the parallelogram which touches an ellipse at the ends of conjugate diameters is constant and equal to the area of the rectangle which touches the ellipse at the ends of the axes.*



Proof. From the last equation we have $a'b' \sin \varphi = ab$; but $a'b' \sin \varphi$ is equal to the area of the parallelogram $CPQD$, and ab is equal to the area of the rectangle $CAEB$; therefore the parallelogram $QRST =$ the rectangle $EFGH$, which is constant.

Cor. 1. The triangle CPD is equal to the triangle ACB, each being one half of the parallelograms QC and EC respectively.

Cor. 2. If P denote the perpendicular from C on QT, we have

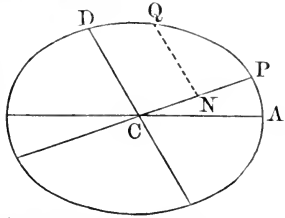
$$P \,.\, CD = \text{area of } CPQD$$
$$= ab.$$

Therefore $$P^2 = \frac{a^2b^2}{CD^2} = \frac{a^2b^2}{b'^2}.$$

But, by § 126, $$b'^2 = a^2 + b^2 - a'^2.$$

Hence $$P^2 = \frac{a^2b^2}{a^2 + b^2 - a'^2}. \qquad (11)$$

129. PROBLEM. *To find the equation of the ellipse referred to a pair of conjugate diameters as axes.*



Let CP, CD be any two conjugate semi-diameters; take CP for the new axis of X, and CD for the new axis of Y; let the angle $ACP = \alpha$, and the angle $ACD = \beta$; (x, y) the co-ordinates of any point Q of the ellipse referred to rectangular axes, and (x', y') the co-ordinates of the same point referred to the new axes.

The formulæ for passing from rectangular to oblique axes are (§ 29)

$$x = x' \cos\alpha + y' \cos\beta;$$
$$y = x' \sin\alpha + y' \sin\beta.$$

But since (x, y) is on the ellipse, we have

$$a^2y^2 + b^2x^2 = a^2b^2.$$

Eliminating x and y from these three equations, we have, after reduction,

$$(a^2 \sin^2\alpha + b^2 \cos^2\alpha)x'^2 + (a^2 \sin^2\beta + b^2 \cos^2\beta)y'^2$$
$$+ 2(a^2 \sin\alpha \sin\beta + b^2 \cos\alpha \cos\beta)x'y' = a^2b^2.$$

But since CP and CD are conjugate semi-diameters, we have, by (8), the condition

$$mm' = -\frac{b^2}{a^2},$$

or
$$\tan\alpha \tan\beta = -\frac{b^2}{a^2};$$

that is,
$$\frac{\sin\alpha \sin\beta}{\cos\alpha \cos\beta} = -\frac{b^2}{a^2},$$

or
$$a^2 \sin\alpha \sin\beta + b^2 \cos\alpha \cos\beta = 0.$$

Therefore the coefficient of $x'y'$ vanishes and we have

$$(a^2 \sin^2\alpha + b^2\cos^2\alpha)x'^2 + (a^2 \sin^2\beta + b^2 \cos^2\beta)y'^2 = a^2b^2, \quad (12)$$

which is the equation of the ellipse referred to the new axes.

By putting $y'^2 = 0$, we get

$$x'^2 = \frac{a^2b^2}{a^2 \sin^2\alpha + b^2 \cos^2\alpha} = CP^2,$$

which we have already denoted by a'^2. In a similar manner we get

$$y'^2 = \frac{a^2b^2}{a^2\sin^2\beta + b^2\cos^2\beta} = CD^2,$$

which we have denoted by b'^2.

Hence (12) may be written

$$\frac{x'^2}{a'^2} + \frac{y'^2}{b'^2} = 1;$$

or, suppressing the accents on the variables, since the equation is entirely general,

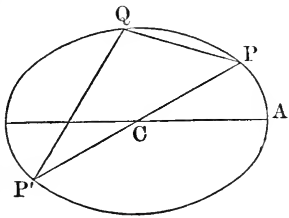
$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1. \tag{13}$$

Comparing this with (1), we see that the equation of the curve referred to the major and minor axes is only a particular form of the more general one which we have just obtained. From the identity of form in (1) and (13) we see that the transformations of the former are applicable to the latter; therefore it follows that any formulæ derived from the equation of the ellipse by processes which do not presuppose the axes to be rectangular will be applicable when any pair of conjugate semi-diameters are substituted for the principal semi-axes.

130. *Supplemental Chords.*

Def. The two straight lines drawn from any point on an ellipse to the extremities of any diameter are called **supplemental chords.**

If the diameter is the major axis, the chords are called **principal supplemental chords.**



Relation between Two Supplemental Chords. Let PP' be any diameter, and PQ, $P'Q$ two supplemental chords; (x', y') the co-ordinates of P, and therefore $(-x', -y')$ the co-ordinates of P', and (x, y) the co-ordinates of Q. Then the equation of the line PQ may be written

$$y - y' = m(x - x'),$$

and the equation of the line $P'Q$ may be written

$$y + y' = m'(x + x');$$

whence, by multiplication,

$$y^2 - y'^2 = mm'(x^2 - x'^2). \qquad (a)$$

But since the points (x, y) and (x', y') are on the curve, we have

$$a^2y^2 + b^2x^2 = a^2b^2$$

and

$$a^2y'^2 + b^2x'^2 = a^2b^2;$$

whence

$$a^2(y^2 - y'^2) + b^2(x^2 - x'^2) = 0,$$

or

$$y^2 - y'^2 = -\frac{b^2}{a^2}(x^2 - x'^2). \qquad (b)$$

Comparing (a) and (b), we have

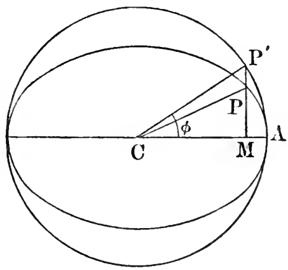
$$mm' = -\frac{b^2}{a^2},$$

which is the condition that holds for conjugate diameters whose slopes to the major axis are m and m' respectively (§ 124); therefore—

THEOREM VIII. *If any chord of an ellipse is parallel to a diameter, the supplemental chord is parallel to the conjugate diameter.*

Relation of the Ellipse and Circle.

131. Let a circle be described on the major axis of an ellipse as a diameter; its equation referred to the centre as origin is



$$y_c^2 = a^2 - x^2, \qquad (a)$$

where y_c represents the ordinate $P'M$.

The equation of the ellipse gives

$$y_e^2 = \frac{b^2}{a^2}(a^2 - x^2). \qquad (b)$$

Comparing (a) and (b), we have

$$\frac{y_c^2}{y_e^2} = \frac{a^2}{b^2}, \quad \text{or} \quad \frac{y_c}{y_e} = \frac{a}{b};$$

whence

$$y_e = \frac{b}{a}y_c;$$

that is, the ordinate of the ellipse at any point is found by multiplying the ordinate of the circle by the constant factor $\frac{b}{a}$. Hence we have

THEOREM IX. *If all the ordinates of a circle be diminished in the same proportion, the circle will be changed into an ellipse.*

132. *The Eccentric Angle.*

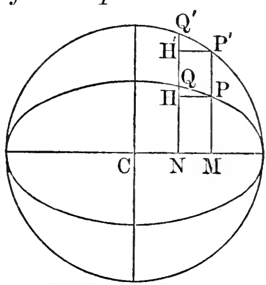
Def. If we join P and C, the centre of the ellipse, *the angle $P'CA$ is called the* **eccentric angle** *of the point P.*

PROBLEM. *To express the co-ordinates of any point of the ellipse in terms of the eccentric angle of that point.*

Let the eccentric angle $= \varphi$, and x, y, the co-ordinates of the point P. Then, since $P'C = AC$, we shall have

$$x = a \cos \varphi;$$
$$y = \frac{b}{a} P'M$$
$$= \frac{b}{a} a \sin \varphi = b \sin \varphi.$$

133. PROBLEM. *To find the area of an ellipse.*



Describe a circle on the major axis as a diameter, which we can conceive to be divided into any number of equal parts. At any two adjacent points, as M, N, draw the common ordinates MP', NQ', and through P and P' draw PH, $P'H'$ parallel to the axis. Let the ordinates PM, $P'M$ be denoted by y_e and y_c respectively. Then, since the rectangles MH, MH' have the same breadth, namely, MN, they are to each other as their heights MP, MP'; that is,

$$\frac{MH}{MH'} = \frac{y_e}{y_c} = \frac{b}{a}. \qquad (\S\ 131)$$

In the same way it may be shown that any other pair of similar rectangles in the ellipse and circle have the ratio of

$b : a$, and therefore the sum of all the rectangles in the ellipse is to the sum of all the corresponding rectangles in the circle as $b : a$.

Now if the number of equal parts into which the axis is divided be increased indefinitely, the sum of all the rectangles in the ellipse will approach the area of the semi-ellipse as a limit, and the sum of all the rectangles in the circle will approach the area of the semi-circle as a limit.

Therefore we shall ultimately have

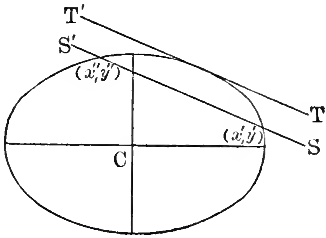
$$\frac{\text{Area of the ellipse}}{\text{Area of the circle}} = \frac{b}{a}.$$

But the area of the circle $= \pi a^2$; therefore we shall have the area of the ellipse $= \pi ab$. Hence:

THEOREM X. *The area of an ellipse is a mean proportional between the areas of the circles described on the major and minor axes.*

Tangents and Normals to an Ellipse.

134. PROBLEM. *To find the equation of the tangent to an ellipse at a given point.*



Let x', y' be the co-ordinates of any point on the curve, and x'', y'' the co-ordinates of an adjacent point on the curve. The equation of the secant passing through the points x', y' and x'', y'' is, by § 45,

$$y - y' = \frac{y'' - y'}{x'' - x'}(x - x'). \qquad (a)$$

Since (x', y') and (x'', y'') are on the ellipse, we have

$$a^2y'^2 + b^2x'^2 = a^2b^2$$

and

$$a^2y''^2 + b^2x''^2 = a^2b^2;$$

therefore

$$a^2(y''^2 - y'^2) + b^2(x''^2 - x'^2) = 0;$$

whence

$$\frac{y'' - y'}{x'' - x'} = -\frac{b^2}{a^2} \cdot \frac{x'' + x'}{y'' + y'}.$$

Substituting in (a), we have, for the equation of the secant,

$$y - y' = -\frac{b^2}{a^2} \cdot \frac{x'' + x'}{y'' + y'}(x - x'). \qquad (b)$$

Now if the points (x', y') and (x'', y'') approach each other until they coincide, the secant SS' will become the tangent TT'. We shall then have at the limit

$$x'' = x' \qquad \text{and} \qquad y'' = y';$$

hence (b) becomes

$$y - y' = -\frac{b^2}{a^2}\frac{x'}{y'}(x - x'),$$

which is the equation of the tangent at the point x', y'.

This equation may be simplified thus: Multiply by a^2y' and we get

$$\begin{aligned} a^2yy' + b^2xx' &= a^2y'^2 + b^2x'^2 \\ &= a^2b^2, \end{aligned}$$

or

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1. \qquad (15)$$

The equation of the tangent may also be expressed independently of the co-ordinates of the point of contact, as follows:

135. Problem. *To find the condition that the line*

$$y = mx + h$$

may be tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

If we eliminate y between these equations, we have

$$\frac{x^2}{a^2} + \frac{(mx + h)^2}{b^2} = 1,$$

or

$$(b^2 + a^2m^2)x^2 + 2a^2mhx = a^2(b^2 - h^2), \qquad (a)$$

for determining the abscissæ of the points in which the line intersects the ellipse. Since the line is to be a tangent to the ellipse, the two values of the abscissa will be equal. Now the

condition that this equation may have equal roots is, by the theory of quadratic equations (§ 8),

$$4a^2(b^2 + a^2m^2)(b^2 - h^2) + 4a^4m^2h^2 = 0$$

whence $$h^2 = b^2 + a^2m^2, \tag{16}$$

or $$h = \pm \sqrt{b^2 + a^2m^2},$$

the required condition.

Substituting this value of h in the given equation of the line, we have

$$y = mx \pm \sqrt{b^2 + a^2m^2} \tag{17}$$

for the equation of the tangent.

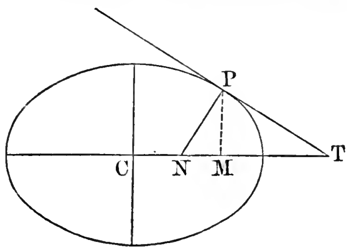
Conversely, every equation of this form is the equation of some tangent to the ellipse. The double sign shows that there will always be two tangents having a given slope.

REMARK. From the facility with which this equation enables us to solve many problems involving the use of the equation of the tangent, it is sometimes called the *magical* equation of the tangent.

136. *The Subtangent.*

Def. The projection on the axis of X of that portion of the tangent intercepted between the point of contact and the axis of X is called the **subtangent.**

To find where the tangent intersects the axis of X, we make $y = 0$ in the equation of the tangent. Thus the equation of the tangent is



$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1.$$

Making $y = 0$, we have

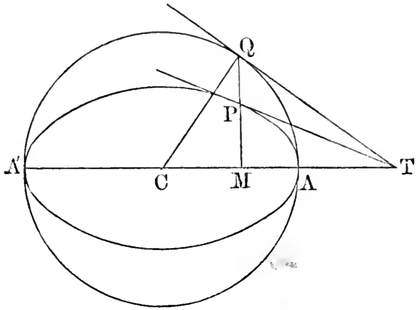
$$x = \frac{a^2}{x'}, = CT.$$

Subtracting CM or x', we have

$$\text{Subtangent} = MT$$
$$= \frac{a^2}{x'} - x' = \frac{a^2 - x'^2}{x'}.$$

Cor. The subtangent is independent of b; hence *all*

ellipses described on a common major axis have a common subtangent for any given abscissa of the points of contact.



This property enables us to draw a tangent to an ellipse from any point on the curve.

Thus, let P be any point on the curve; describe a circle on AA' as a diameter, and produce the ordinate PM to meet the circle in Q. Then if x' is the abscissa CM, we have

$$\text{Subtangent of ellipse} = \frac{a^2 - x'^2}{x'} = \text{subtangent of circle} = MT.$$

Hence if QT be drawn tangent to the circle and meeting AA' produced in T, then, by what has just been proved, T will be the foot of tangent to the ellipse at P, which is found by joining TP. If the point T were given, we would first draw TQ tangent to the circle, and from the point of contact Q draw the ordinate QM, intersecting the ellipse in P, the required point of contact; and by joining P and T we would have the required tangent.

137. *Tangent through a Given Point.*

Let the tangent line be required to pass through a given point (x', y'); we shall then have the condition

$$y' = mx' + h, \tag{a}$$

which, combined with (16), will enable us to determine m and h. Equation (a) gives

$$h^2 = y'^2 - 2mx'y' + m^2x'^2,$$

which, substituted in (16), gives

$$(a^2 - x'^2)m^2 + 2x'y'm + b^2 - y'^2 = 0;$$

whence $$m = \frac{-x'y' \pm \sqrt{a^2y'^2 + b^2x'^2 - a^2b^2}}{a^2 - x'^2}. \tag{18}$$

Since there are *two* values of m, two tangents to an ellipse can be drawn through a given point. There are three cases depending on the position of the point:

I. If the position of the point is such that

$$a^2y'^2 + b^2x'^2 - a^2b^2 < 0,$$

the value of m will be imaginary. The point (x', y') will then be within the ellipse.

II. If $a^2y'^2 + b^2x'^2 - a^2b^2 > 0$, the two values will be *real* and different.

III. If $a^2y'^2 + b^2x'^2 - a^2b^2 = 0$, the point (x', y') will be on the ellipse, the two tangents will coincide, and the equation can be reduced to the form (16).

138. PROBLEM. *To find the locus of the point from which two tangents to an ellipse make a right angle with each other.*

Let the equations of the tangents be

$$y = mx + \sqrt{b^2 + a^2m^2}; \tag{a}$$

$$y = m'x + \sqrt{b^2 + a^2m'^2}. \tag{b}$$

Then the condition to be fulfilled is (§ 47)

$$mm' + 1 = 0. \tag{c}$$

Eliminating m' from (b) and (c), the equation of the two tangents will be

$$y - mx = \sqrt{b^2 + a^2m^2};$$

$$my + x = \sqrt{a^2 + b^2m^2}.$$

Now, what we want is the locus of the point which is on both tangents at once; that is, the locus of the point whose co-ordinates satisfy both of these equations. To find the required locus, we must eliminate m from the equations, which we do thus:

Squaring and adding, we have

$$(m^2+1)x^2+(m^2+1)y^2=(m^2+1)(a^2+b^2),$$

or

$$x^2+y^2=a^2+b^2,$$

which is the equation of a circle whose centre is at the origin and whose radius is $\sqrt{a^2+b^2}$.

We thus have the result: *If we slide a right angle around an ellipse so that its sides shall continually touch the ellipse, its vertex will describe a circle whose radius is equal to the distance between the ends of the major and minor axes.*

139. PROBLEM. *A perpendicular being drawn from either focus of an ellipse upon a moving tangent, it is required to find the locus of the foot of the perpendicular.*

Let

$$y=mx+\sqrt{b^2+a^2m^2} \tag{a}$$

be the equation of the tangent. The equation of a line perpendicular to (a) and passing through the focus whose coordinates are ae and 0 is

$$y=-\frac{1}{m}(x-ae). \tag{b}$$

From (a) we have

$$y-mx=\sqrt{b^2+a^2m^2},$$

and from (b),

$$my+x=ae.$$

Squaring and adding, we get

$$\begin{aligned}(x^2+y^2)(1+m^2)&=b^2+a^2m^2+a^2e^2\\&=a^2(1+m^2)\end{aligned}$$

(since $a^2e^2+b^2=a^2$).

Therefore we have

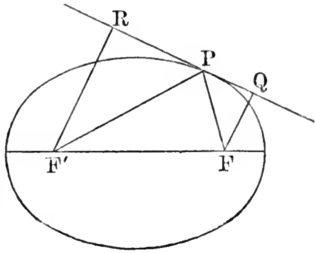
$$x^2+y^2=a^2,$$

the equation of the required locus, which is a circle described on the major axis of the ellipse. The same result is obtained if we draw the perpendicular from the other focus.

140. *Perpendiculars from the Foci upon the Tangent.*

PROBLEM. *To find an expression for the length of the per-*

pendicular from either focus upon the tangent to an ellipse at the point (x', y').



Let p and p' be the perpendiculars FQ, $F'R$ respectively. The equation of the tangent is

$$b^2x'x + a^2y'y - a^2b^2 = 0;$$

and since the co-ordinates of the foci F and F' are $(ae, 0)$ and $(-ae, 0)$ respectively, we shall have, by § 41,

$$\left.\begin{aligned} p &= \frac{ab^2ex' - a^2b^2}{\sqrt{b^4x'^2 + a^4y'^2}} = \frac{ab^2(ex' - a)}{\sqrt{b^4x'^2 + a^4y'^2}} \\ \text{and}\qquad p' &= \frac{-ab^2ex' - a^2b^2}{\sqrt{b^4x'^2 + a^4y'^2}} = \frac{-ab^2(ex' + a)}{\sqrt{b^4x'^2 + a^4y'^2}}, \end{aligned}\right\} \qquad (19)$$

which are the required expressions for the perpendiculars.

Product of the Perpendiculars from the Foci upon the sam Tangent. We find, by multiplication,

$$\begin{aligned} pp' &= \frac{a^2b^4(a^2 - e^2x'^2)}{b^4x'^2 + a^4y'^2} \\ &= \frac{a^2b^4(a^2 - e^2x'^2)}{b^4x'^2 + a^2(a^2b^2 - b^2x'^2)} \\ &= \frac{a^2b^2(a^2 - e^2x'^2)}{b^2x'^2 + a^2(a^2 - x'^2)} \\ &= \frac{a^4(1 - e^2)(a^2 - e^2x'^2)}{a^2(1 - e^2)x'^2 + a^2(a^2 - x'^2)} \\ &\qquad\qquad [\text{since } b^2 = a^2(1 - e^2)] \\ &= \frac{a^2(1 - e^2)(a^2 - e^2x'^2)}{a^2 - e^2x'^2} \\ &= b^2, \end{aligned} \qquad (20)$$

an expression which is independent of the co-ordinates x' and y'.

Hence:

THEOREM XI. *The rectangle contained by the perpendiculars from the foci upon a tangent to an ellipse is constant and equal to the square of the semi-minor axis.*

For the ratio of the perpendiculars we have

$$\frac{p}{p'} = \frac{a - ex'}{a + ex'}$$
$$= \frac{r}{r'}. \qquad (\S\ 120)$$

Hence:

THEOREM XII. *The perpendiculars from the foci upon the tangent have to each other the same ratio as the focal radii of the point of tangency.*

141. *The Normal to an Ellipse.*

PROBLEM. *To find the equation of the normal line at any point of an ellipse.*

Let x', y' be the co-ordinates of any point on the ellipse. Then, by § 134, the equation of the tangent at that point is

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1, \qquad (a)$$

or

$$y = -\frac{b^2x'}{a^2y'}x + \frac{b^2}{y'}.$$

The equation of a line through x', y' and perpendicular to (a) is, by § 47,

$$y - y' = \frac{a^2y'}{b^2x'}(x - x'), \qquad (21)$$

which is the equation of the normal at x', y'.

142. *The Subnormal.*

Def. That portion of the normal line intercepted between the point on the curve and the axis of X is called *the normal*, and its projection on the axis of X is called *the subnormal.*

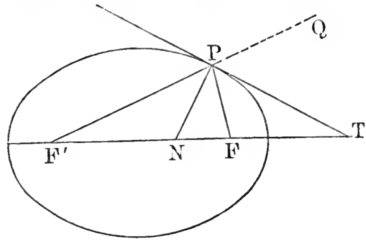
To find where the normal cuts the axis of X, we make $y = 0$ in the equation of the normal; then we get (see fig., § 136)

$$CN = x'\left(\frac{a^2 - b^2}{a^2}\right) = e^2x'.$$

Hence the subnormal

$$NM = CM - CN$$
$$= x' - x'\left(1 - \frac{b^2}{a^2}\right) = \frac{b^2}{a^2}x'$$
$$= (1 - e^2)x'.$$

143. THEOREM XIII. *The normal at any point on an ellipse bisects the angle contained by the focal radii of that point.*



Proof. Let us put ψ, ψ', the angles FPN and $F'PN$ respectively.

By the theorem of sines, we have

$$\frac{\sin\psi}{\sin PNF}=\frac{FN}{FP};\qquad \frac{\sin\psi'}{\sin PNF'}=\frac{F'N}{F'P}. \qquad (a)$$

Now, FP and $F'P$ are the focal radii whose lengths are given by the equations (4) and (5), § 120. Also, by §§ 136 and 142, we readily find

$$FN = ae - e^2x' = e(a - ex') = er;$$
$$F'N = ae + e^2x' = e(a + ex') = er';$$

whence (a) gives

$$e\sin PNF = \sin\psi; \qquad e\sin PNF' = \sin\psi';$$

and then, since $\sin PNF' = \sin PNF$, we have

$$\psi = \psi'.$$

Therefore the normal PN bisects the angle FPF'.

Cor. The tangent at any point of an ellipse bisects the exterior angle formed by the focal radii of that point.

For if one of the focal radii, as $F'P$, be produced to any point Q, and the tangent PT be drawn, the angles $F'PF$, FPQ are supplementary; and since NPT is a right angle and PN bisects the angle $F'PF$, PT also bisects the angle FPQ, which is the exterior angle formed by the focal radii FP, $F'P$.

REMARK. If a ray of light proceed from F to any point P on the ellipse, it will be reflected to F'. For this reason the points F and F' are called *foci*, or burning points.

The theorem just proved enables us to draw a tangent at any point on an ellipse. Thus, let P be any point on the curve; draw the focal radii PF, PF'; produce one of them, as PF', and bisect the exterior angle thus formed by PT, which is the tangent required.

EXERCISES.

1. Show that there is a certain segment of the major axis of an ellipse from all points of ~~which~~ which normals not coincident with that axis may be drawn to the ellipse, and two other segments from which such normals cannot be drawn, and define these segments.

2. Show that the normals from three or more equidistant points on the major axis intersect the ellipse in points whose abscissæ are in arithmetical progression.

3. Show that the ordinate of the point in which a normal intersects the minor axis is in the constant ratio $\dfrac{e^2}{e^2-1}$ to that of the point where it intersects the ellipse.

Reciprocal Polar Relations.

144. *Chord of Contact.*

Def. The line which passes through the points where two tangents from an external point meet an ellipse is called the **chord of contact.**

Problem. *To find the equation of the chord of contact.*

Let (h, k) be the co-ordinates of the point from which the two tangents are drawn; (x', y'), the co-ordinates of the point where one of the tangents through (h, k) meets the curve, and (x'', y'') the co-ordinates of the point where the other tangent meets the curve.

The equation of the tangent at (x', y') is

$$\frac{x'x}{a^2} + \frac{y'y}{b^2} = 1; \qquad (a)$$

and since this passes through (h, k), we have

$$\frac{hx'}{a^2} + \frac{ky'}{b^2} = 1. \qquad (b)$$

Similarly,

$$\frac{hx''}{a^2} + \frac{ky''}{b^2} = 1. \qquad (c)$$

Hence it follows that the equation of the chord of contact is

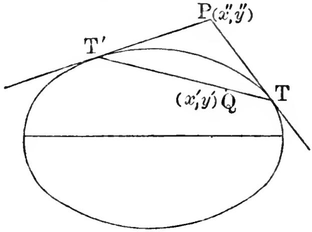
$$\frac{hx}{a^2} + \frac{ky}{b^2} = 1, \qquad (22)$$

for this is the equation of a straight line, and is satisfied for $x = x'$, $y = y'$ and $x = x''$, $y = y''$, as we see from (b) and (c).

Cor. From what has been shown in the preceding section, it is evident that this equation, referred to any pair of conjugate diameters as axes, is

$$\frac{hx}{a'^2} + \frac{ky}{b'^2} = 1. \tag{23}$$

145. *Locus of Intersection of Two Tangents.* Let (x', y') be the co-ordinates of any fixed point Q through which the chord of contact corresponding to the two intersecting tangents is drawn; (x'', y''), the co-ordinates of P, the intersection of the tangents. By the preceding section, the equation of the chord TT' is



$$\frac{x''x}{a^2} + \frac{y''y}{b^2} = 1;$$

but since (x', y') is a point on the chord, we have the condition

$$\frac{x''x'}{a^2} + \frac{y''y'}{b^2} = 1,$$

which the co-ordinates of the point of intersection must always satisfy. Hence, regarding x'', y'' as *variables* and omitting the accents, the equation of the locus of the point of intersection of the two tangents is

$$\frac{x'x}{a^2} + \frac{y'y}{b^2} = 1. \tag{24}$$

Cor. This equation, referred to a pair of conjugate diameters as axes, will be

$$\frac{x'x}{a'^2} + \frac{y'y}{b'^2} = 1. \tag{25}$$

146. *Pole and Polar.*

The identity of form in the equations of the *tangent*, the *chord of contact* and the *locus* of the intersection of tangents

drawn from the extremities of chords passing through a fixed point is only the expression of a reciprocal relation which exists between the *locus* and the fixed point (x', y'). This relation is one of polar reciprocity and is expressed by the following theorem:

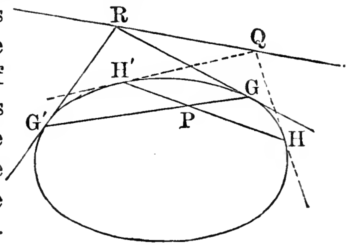
THEOREM XIV. 1. *If chords in an ellipse be drawn through any fixed point and tangents be drawn from the extremities of each chord, the locus of the intersections of the several pairs of tangents will be a straight line.*

2. Conversely, *If from different points in a straight line pairs of tangents be drawn to an ellipse, their chords of contact will intersect in one point.*

Defs. The straight line which forms the locus of the intersection of two tangents drawn from the extremities of any chord which passes through a fixed point is called the **polar** of that point.

Reciprocally, the fixed point is called the **pole** of the straight line which forms the locus.

Thus, if P be the fixed point through which the chords GG', HH' are drawn, and pairs of tangents GR, $G'R$, HQ, $H'Q$ be drawn from their extremities, intersecting in R and Q respectively, then the line QR is the *polar* of P, and P is the pole of QR. If the pole is on the curve as at H, then the tangent HR is the *polar;* and if the pole is *without* the curve, as at Q, then it follows that the chord of contact HH' is the polar; hence we see that the *tangent* and the *chord of contact* are respectively the polars of the point of contact and of the intersection of the tangents drawn from the extremities of the chord of contact.



Hence it follows that if (x', y') be the co-ordinates of any point *within, on* or *without* the curve, the equation of the *polar* is

$$\frac{x'x}{a^2} + \frac{y'y}{b^2} = 1, \qquad (26)$$

or, when referred to a pair of conjugate diameters as axes,

$$\frac{x'x}{a'^2} + \frac{y'y}{b'^2} = 1. \tag{27}$$

The equation of the diameter conjugate to that which passes through the point (x', y') is

$$y = -\frac{b'^2}{a'^2}\frac{x'}{y'}x,$$

or

$$\frac{x'x}{a'^2} + \frac{y'y}{b'^2} = 0,$$

which shows that the diameter and the polar (27) are parallel; hence we have the following theorem:

The polar of any point in respect to an ellipse is parallel to the diameter conjugate to that which passes through the point.

147. *Polars of Special Points.*

Polar of the Centre. If in the equation of the polar (26) we suppose the pole (x', y') to approach the centre, x' and y' will approach zero as their limit, and one or both the co-ordinates, x and y, of any point of the polar will increase indefinitely. Hence *the polar of the centre is at infinity.*

This is also seen from the fact that tangents at the extremities of any diameter meet at infinity.

Polar of a Point on one of the Axes. When $y' = 0$, we get

$$x = \frac{a^2}{x'} = \text{a constant},$$

which shows that the semi-major axis is a mean proportional between the distances, x' and x, of the pole and polar from the centre. Since the same reasoning may be applied to a point on the minor axis, we conclude:

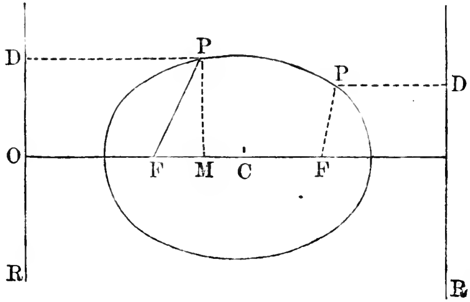
THEOREM XV. *Either semi-axis is a mean proportional between the distances cut off from it by a pole upon it, and by the corresponding polar.*

Polar of the Focus. Substituting for (x', y') the co-ordinates of either focus $(\pm\, ae, 0)$ in (26), we have

$$x = \pm\, \frac{a}{e};$$

or, *the polar of either focus of an ellipse is perpendicular to the major axis and at a distance from the centre equal to* $\frac{a}{e}$ *measured on the same side as the focus.*

148. *Directrix of an Ellipse.*



If DR is the polar of the focus F, we have

$$OC = \frac{a}{e},$$

and

$$\begin{aligned} DP &= OM \\ &= OC - MC \\ &= \frac{a}{e} - x = \frac{a - ex}{e}; \end{aligned}$$

but from the linear equation of the curve we have

$$FP = a - ex;$$

hence $$DP = \frac{FP}{e} \quad \text{and} \quad \frac{FP}{DP} = e.$$

The same reasoning applies to either focus and its polar. Hence:

THEOREM XVI. *The focal distance of any point on an ellipse is in a constant ratio to its distance from the polar of the corresponding focus, the ratio being less than unity and equal to the eccentricity of the curve.*

Def. The polar of either focus is called a **directrix**.

EXERCISES.

1. Show that an ellipse has a pair of equal conjugate diameters whose direction coincides with the diagonals of the rectangle on the axes.

2. Show that the equal conjugate diameters of an ellipse bisect the lines joining the extremities of the axes.

3. Find the co-ordinates of the point in an ellipse such that the tangent there is equally inclined to the axes.

$$Ans.\ \frac{a^2}{\sqrt{a^2+b^2}}; \qquad \frac{b^2}{\sqrt{a^2+b^2}}.$$

4. If r and r' denote the focal radii of any point on an ellipse whose eccentric angle is φ, show that

$$r = a(1 - e\cos\varphi) \quad \text{and} \quad r' = a(1 + e\cos\varphi).$$

5. Find the equation of the tangent at the extremity of the latus rectum. *Ans.* $y + ex = a$.

6. Find the equations of the lines joining (1) the extremities of the axes; (2) the centre and the extremities of the latera recta.

$$Ans.\ y = \pm\frac{b}{a}\left(x \mp a\right); \quad y = \pm\frac{b^2}{a^2}\cdot\frac{x}{e}.$$

7. Find the equation of the normal at the extremity of the latus rectum.

$$Ans.\ y - \frac{x}{e} + ae^2 = 0.$$

8. If the normal at the extremity of the latus rectum passes through the extremity of the minor axis, show that the eccentricity of the ellipse is determined by the equation

$$e^4 + e^2 - 1 = 0.$$

9. Show that the equation of the tangent at any point is

$$\frac{x}{a}\cos\varphi + \frac{y}{b}\sin\varphi - 1 = 0,$$

where φ is the eccentric angle of that point.

10. Find the equation of the straight line which is tangent to the ellipse $20y^2 + 5x^2 = 100$ at the point (2, 2).

11. Through the right-hand focus of the ellipse $25y^2 + 16x^2 = 1600$ is drawn a focal radius making an angle of 30° with the axis of X. Find the equation of the tangent to the ellipse at the end of this radius.

12. Express the intercepts which the normal to an ellipse cuts off from the co-ordinate axes in terms of the principal axes of the ellipse and of the co-ordinates of the point (x_1, y_1) in which the normal cuts the ellipse.

13. If θ is the angle which a radius from the centre of the ellipse forms with the axis of X, and θ' the angle which the tangent to the ellipse at the end of that radius forms with the same axis, find what relation exists between θ and θ'. $\tan\theta \tan\theta' =$

14. From the centre of an ellipse to a tangent is drawn a line parallel to the focal radius of the point of tangency, and meeting the tangent at the point p. Find the locus of p as the tangent changes its position.

15. From one focus of an ellipse a perpendicular is dropped upon the tangent and produced to an equal distance on the other side. Show that its terminus is in the same straight line with the point of tangency and the other focus. By Cor. of Th. ...

16. The same thing being supposed, find the locus of p when the tangent moves around the ellipse.

17. To the ellipse $a^2y^2 + b^2x^2 = a^2b^2$ and its circumscribing circle $y^2 + x^2 = a^2$ tangents are drawn such that the points of tangency shall have the same abscissa. What relation exists between the subtangents, and what relation between the subnormals?

18. Find the equations of the tangents drawn from the point (0, 8) to the ellipse whose equation is $20y^2 + 5x^2 = 100$.

19. If that point of an ellipse to which a normal is drawn approaches indefinitely near to the major axis, what limit will the intercept of the normal upon the axis of X approach?

20. On the major axis of an ellipse a point is taken whose abscissa is x'. Find the slope and equation of the tangents from this point.

21. At what points will the tangents which make an angle of 45° with the principal axes cut those axes?

22. Find the intercept upon the minor axis when the normal approaches the end of that axis.

23. Find the equations of the two tangents to the ellipse $5y^2 + 3x^2 = 15$ which are parallel to the line $3y - 4x + 1 = 0$.

24. To the ellipse $36y^2 + 25x^2 = 900$ are to be drawn tangents cutting the axis of X at an angle of 30°. Find the co-ordinates of the points of tangency.

25. Having given the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ and the circle $x^2 + y^2 = ab$, it is required to find the equation of the common tangent to the two curves. Find also the angle at which the curves intersect.

26. If two points as poles be taken on a tangent to an ellipse, where will their polars intersect?

27. The chord of contact to two tangents of an ellipse is required to pass through the focus. What is the locus of the point where the tangents intersect?

28. Find the pole of the line $y = mx + h$ with respect to the ellipse $a^2y^2 + b^2x^2 = a^2b^2$.

29. If tangents to the circumscribed circle of an ellipse be taken as polars, what will be the locus of the pole?

30. Find the locus of the pole when the polar is required to be a tangent to the circle described upon the minor axis of the ellipse as a diameter.

31. If a series of poles be taken on the diameter of an ellipse, show that the polars will all be parallel to each other.

32. If chords be drawn from any point of an ellipse to the ends of either principal axis, show geometrically that they are parallel to a pair of conjugate diameters.

33. If a line of fixed length slide with its two ends constantly upon the respective sides of a right angle, show that any point upon it describes an ellipse.

34. The area of an ellipse is to be equal to that of the concentric circle passing through its foci. Find its eccentricity.

Ans. $e = \left\{ \dfrac{\sqrt{5} - 1}{2} \right\}^{\frac{1}{2}}$.

35. The minor axis of an ellipse is 12, and its area is equal to that of a circle whose diameter is 20. What is its major axis?

36. The area of an ellipse is equal to that of a circle circumscribed around the square upon its minor axis. Find the angle whose sine is the eccentricity. *Ans.* 60°.

37. Show that the equation of the normal at the point whose eccentric angle is φ is

$$ax \sec \varphi - by \operatorname{cosec} \varphi = a^2 - b^2.$$

38. If φ and φ' be the eccentric angles of any two points P, Q on an ellipse, show that the area of the parallelogram formed by tangents at the extremities of the diameters through P and Q is $\dfrac{4ab}{\sin(\varphi' - \varphi)}$. When is this area a minimum?

39. Show that the circle described on any focal chord as a diameter touches the circle described on the major axis as a diameter. Err

40. Normals are drawn to an ellipse and to the circumscribing circle at points having the same abscissa. Show that the locus of their intersection is a circle whose radius is $a + b$.

41. Show that the locus of the intersection of tangents to an ellipse at the extremities of conjugate diameters is an ellipse.

42. Show that the tangents at the extremities of any chord of an ellipse meet on the diameter which bisects that chord.

43. If φ and φ' denote the eccentric angles of the vertices of two conjugate diameters of an ellipse, show that

$$\tan \varphi \tan \varphi' + 1 = 0.$$

44. If θ denote the angle which any focal chord makes with the major axis, show that the length of the chord is $\dfrac{2b^2}{a(1 - e^2 \cos^2 \theta)}$, and the length of the diameter parallel to the chord is $\dfrac{2b}{\sqrt{(1 - e^2 \cos^2 \theta)}}$.

45. If φ and φ' be the eccentric angles of any two points

on an ellipse, show that the equation of the chord which joins the points is

$$b \cos \tfrac{1}{2}(\varphi + \varphi') \cdot x + a \sin \tfrac{1}{2}(\varphi + \varphi') \cdot y = ab \cos \tfrac{1}{2}(\varphi - \varphi').$$

46. Find the polar equation of the ellipse (1) when the centre is the pole, and (2) when the left-hand vertex is the pole, the major axis being the initial line in both cases.

$$\textit{Ans.}\ r^2 = \frac{b^2}{1 - e^2 \cos^2\theta}; \qquad r = \frac{2ab^2 \cos\theta}{a^2 \sin^2\theta + b^2 \cos^2\theta}.$$

47. Show that the perpendicular from the centre on the chord which joins the extremities of two perpendicular diameters of an ellipse is of constant length.

48. Find the polar co-ordinates of that point on an ellipse at which the angle between the radius vector and tangent is a minimum. *Ans.* a; $\cos^{-1}e$.

49. If the equation $x^2 + y^2 = a^2$ represent an ellipse, express its eccentricity in terms of the angle between the axes.

50. Show that the sum of the reciprocals of two focal chords at right angles to each other is constant and equal to

$$\frac{a^2 + b^2}{2ab^2}.$$

51. A tangent is inclined to the major axis of an ellipse at an angle θ. Show that the rectangle contained by perpendiculars upon it from the ends of the major axis varies as $\cos^2\theta$.

52. If r_1, r, r_2 be the radii vectores corresponding to the angles $\theta - 60°$, θ, $\theta + 60°$, show that

$$\frac{1}{r_1} + \frac{1}{r_2} - \frac{1}{r} = \frac{1}{\frac{1}{2}\text{ latus rectum}}.$$

53. Show from the equation $y^2 = \frac{b^2}{a^2}(2ax - x^2)$ and from § 119 that if the major axis of an ellipse becomes infinite while the parameter remains finite, the ellipse will become a parabola.

54. Show that the line from the focus to the point of intersection of two tangents bisects the angle formed by the focal radii of the points of tangency.

CHAPTER VII.

THE HYPERBOL

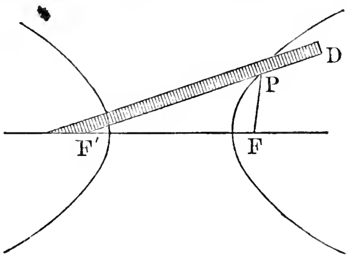
Equation and Fundamental Properties of the Hyperbola.

149. *Def.* An **hyperbola** is the locus of a point the difference of whose distances from two fixed points is constant.

The two fixed points are called the **foci** of the hyperbola.

The distances from any point on the curve to the foci are called **focal radii,** or *focal distances.*

The hyperbola is described mechanically as follows: Take any two fixed points, as F and F', and at one of them, as F', let a ruler be pivoted, while to the other point, F, is fastened a thread whose length is *less* than that of the ruler.



Attach the other end of the thread to the free end of the ruler at D, and stretch the thread tightly against the edge of the ruler with a pencil-point, P. Then, while the ruler is moved round the pivot at F', let the pencil-point slide along the edge of the ruler so as to keep the thread tightly stretched; the pencil-point will describe an *hyperbola*, because in every position of P we shall have

$$F'P - FP = (F'P + PD) - (FP + PD).$$

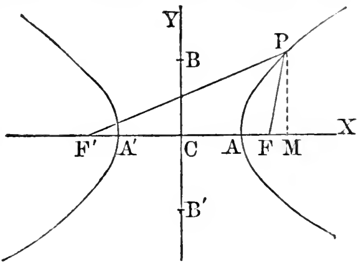
But $F'P + PD$ is the length of the ruler, and $FP + PD$ is the length of the thread, and the difference between the lengths of these is constant; therefore we have

$$F'P - FP = \text{a constant},$$

which agrees with the definition.

By interchanging the fixed extremities of the ruler and thread we shall obtain a second figure equal and similar in every respect to the first, but turned in the opposite direction. Thus we see that the complete curve consists of *two* branches, as represented above.

150. PROBLEM. *To find the equation of the hyperbola.*



Let the straight line drawn through the foci, be taken as the axis of X; the point C *midway* between the foci be taken as the *origin*, and the perpendicular to FF' through C as the axis of Y. Let the distance between the foci $= 2c$; the difference between any two focal radii $= 2a$; and x, y, the co-ordinates of any point P. Then we have

$$F'M = x + c;$$
$$FM = x - c;$$

and therefore

$$PF'^2 = (x + c)^2 + y^2;$$
$$PF^2 = (x - c)^2 + y^2;$$

and, by the fundamental property of the curve,

$$\sqrt{(x + c)^2 + y^2} - \sqrt{(x - c)^2 + y^2} = 2a.$$

Freeing this equation of surds, we have

$$(c^2 - a^2)x^2 - a^2y^2 = a^2(c^2 - a^2), \qquad (1)$$

which is the required equation.

This, however, may be simplified by putting, for the sake of brevity,

$$c^2 - a^2 = b^2; \qquad (2)$$

hence we have

$$b^2x^2 - a^2y^2 = a^2b^2, \qquad (3)$$

or, dividing through by a^2b^2,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \qquad (4)$$

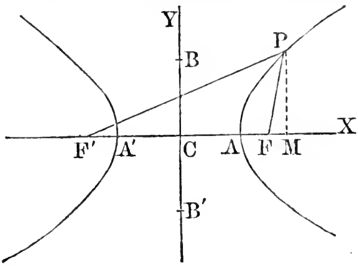
which is the equation of the hyperbola referred to its centre and axes.

151. *Relations among Axes and Foci of the Hyperbola.*

If, in the equation (3), we put $y = 0$, we have, for the points in which the curve cuts the axis of X,

$$x = \pm a = CA \text{ or } CA'.$$

Therefore the curve cuts the axis of X in two points, A and A', equidistant from the origin and between the origin and the foci.



Def. The points A, A' where the line joining the foci cuts the curve are called the **vertices** of the hyperbola.

The line AA' is called the **transverse** axis.

The point C midway between the vertices is called the **centre** of the curve.

If $x = 0$, we have

$$y = \pm b\sqrt{-1},$$

which shows that the curve cuts the axis of Y in two *imaginary* points situated on opposite sides of the centre and at the imaginary distance $b\sqrt{-1}$ from it.

Measure off now on the axis of Y the distances CB, CB', each equal to b, the real factor of this imaginary value of y. Then:

Def. The line BB' is called the **conjugate axis** of the hyperbola.

Solving the equation (3), for y, gives

$$y = \pm \frac{b}{a}\sqrt{x^2 - a^2}, \tag{5}$$

which is real for all values of x greater than a. Hence, when $x > a$, y has two real values equal in magnitude but of opposite signs; therefore the curve is symmetrical in reference to the axis of X.

If $x < a$, the values of y are imaginary; therefore no point of the curve lies nearer to the centre than the vertices.

If x increases without limit in either direction, y increases without limit, and therefore the curve extends indefinitely both to the right and to the left of the points A, A'.

By solving the equation of the curve for x, we can easily show in a similar manner that the curve is symmetrical in reference to the axis of Y.

Def. The distance $CF = CF' = c$ of each focus from the centre is called the **linear eccentricity** of the hyperbola.

The ratio $\frac{c}{a}$ is called the **eccentricity** of the hyperbola, and is represented by the symbol e.

Since $$c^2 = a^2 + b^2,$$
we have $$e^2 = \frac{a^2 + b^2}{a^2}. \qquad (6)$$

Hence the eccentricity of an hyperbola is always greater than *unity*.

From (6) we find
$$b = a\sqrt{e^2 - 1}, \qquad (7)$$
which expresses the semi-conjugate axis in terms of the semi-transverse axis and the eccentricity; and since $e > 1$, b may be *greater* or *less* than a. For this reason we do not use the terms *major* and *minor* axis as in the case of the ellipse.

Cor. By comparing the equation of the ellipse with that of the hyperbola, we see that the equation of the latter may be deduced from that of the former by simply writing $-b^2$ for $+b^2$. Hence

Any function of b in the ellipse will be converted into the corresponding function in the hyperbola by merely changing b into $b\sqrt{-1}$.

152. *Equilateral Hyperbola.* An hyperbola in which the transverse and conjugate axes are equal is called an **equilateral hyperbola.**

From (3) we see that the equation of the equilateral hyperbola is
$$x^2 - y^2 = a^2.$$

153. *Def.* The **parameter** or *latus rectum* of an hyperbola is the chord through the focus perpendicular to the transverse axis.

THEOREM I. *The parameter of an hyperbola is a third proportional to the transverse and conjugate axes.*

In order to find the value of the parameter or latus rectum, we put $x = c$ in the equation of the curve. The equation of the curve may be written

$$y^2 = \frac{b^2}{a^2}\left(x^2 - a^2\right).$$

And substituting c for x and denoting the semi-parameter by p, we have

$$p^2 = \frac{b^2}{a^2}\left(c^2 - a^2\right);$$

whence

$$p = \frac{b^2}{a}, \qquad (8)$$

or $$ap = b^2;$$

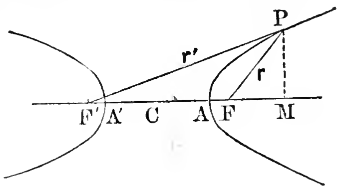
that is, $$a : b :: b : p.$$

Cor. The length of the semi-parameter, in terms of a and e, is

$$p = a(e^2 - 1).$$

154. *Focal Radii.*

PROBLEM. *To express the lengths of the focal radii in terms of the abscissa of the point from which they are drawn.*



Let r and r' denote the focal radii of any point P whose coordinates are (x, y). Then, from the figure. we have

$$\begin{aligned} r^2 &= FM^2 + PM^2 \\ &= (x - ae)^2 + y^2 \\ &= (x - ae)^2 + \frac{b^2}{a^2}\left(x^2 - a^2\right) \\ &= (x - ae)^2 + (e^2 - 1)(x^2 - a^2) \\ &= e^2x^2 - 2aex + a^2; \end{aligned}$$

whence $$r = ex - a. \qquad (9)$$

In a similar manner we find

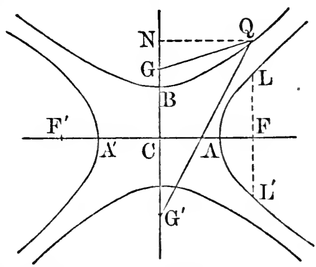
$$r' = ex + a. \tag{10}$$

Either of these expressions, being of one dimension in x, is called the *linear* equation of the hyperbola.

We observe that their difference is $2a$, as it should be.

155. *Conjugate Hyperbola.*

We will now point out the signification of the line BB', whose length is $2b$ and which is defined as the *conjugate* axis of the curve. It is so called by reason of the important relation it bears to a companion-curve, called the **conjugate hyperbola,** whose equation we will now develop.



Let an hyperbola be described about the foci G, G' situated on the axis of Y, and at the same distance from the centre C as the foci F, F' of the hyperbola which we have hitherto been considering. Let (x, y) be the co-ordinates of any point Q on this new curve. Then, retaining the same origin and axes of reference as before, we shall have

$$CG = CG' = c, \qquad x = NQ \qquad \text{and} \qquad y = CN;$$

therefore
$$G'Q^2 = (c + y)^2 + x^2;$$
$$GQ^2 = (c - y)^2 + x^2.$$

Let the difference between the focal radii $G'Q$, GQ be $2b$ instead of $2a$. Then we shall have, by the definition of the curve,

$$\sqrt{(c + y)^2 + x^2} - \sqrt{(c - y)^2 + x^2} = 2b,$$

which, when freed from radicals, becomes

$$b^2x^2 - (c^2 - b^2)y^2 = -b^2(c^2 - b^2).$$

But
$$c^2 - b^2 = a^2;$$

therefore
$$b^2x^2 - a^2y^2 = -a^2b^2,$$

or
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1, \tag{11}$$

which is the equation of the companion-curve or *conjugate hyperbola.*

If in (11) we put $x = 0$,

$$y = \pm\, b = CB \quad \text{or} \quad CB',$$

which shows that the conjugate hyperbola has its transverse axis coinciding in direction and equal in magnitude to the conjugate axis of the primary curve.

If $y = 0$, we have

$$x = \pm\, a\sqrt{-1}.$$

But $CA = a$ and $AA' = 2a$; therefore the transverse axis of the primary curve is the conjugate axis of the new curve. Thus we see that what is called the *conjugate* axis of an hyperbola is in fact the *transverse* axis of the conjugate hyperbola.

Def. A **conjugate hyperbola** is one which has the conjugate axis of a given hyperbola for its transverse axis, and the transverse axis of the given hyperbola for its conjugate axis.

By comparing (4) and (11) we see that the equations of an hyperbola and its conjugate differ only in the sign of the constant term. Since the conjugate hyperbola holds the same relation to the axis of Y that the original does to the axis of X, we may obtain the equation of the former from that of the latter by simply interchanging the quantities which relate to the two axes. Thus if the equation of the original hyperbola is

$$b^2x^2 - a^2y^2 = a^2b^2,$$

then, by interchanging

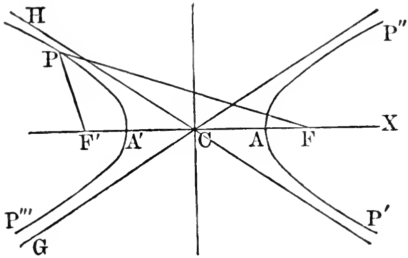
$$x \text{ and } y,$$
$$a \text{ and } b,$$

we have, after changing signs,

$$b^2x^2 - a^2y^2 = -\,a^2b^2,$$

which is the equation of the conjugate hyperbola.

156. *Polar Equation of the Hyperbola, the left-hand focus being the pole, and the transverse axis the initial line.*



Let the angle $A'F'P = \theta$ and $PF' = r$ be the polar coordinates of any point P. Then we shall have

$$PF^2 = PF'^2 + FF'^2 - 2PF'.\ FF' \cos FF'P,$$

or $$PF^2 = r^2 + 4a^2e^2 - 4aer \cos \theta.$$

Now, by the fundamental property of the curve, we have

$$PF - PF' = 2a,$$

or $$\sqrt{r^2 + 4a^2e^2 - 4aer \cos \theta} - r = 2a;$$

whence we get

$$r = \frac{a(e^2 - 1)}{1 + e \cos \theta}, \tag{12}$$

which is the equation required.

The polar equation may also be very readily obtained from the *linear* equation of the curve in the same manner as in the case of the ellipse. (See § 121, Ellipse.)

157. *To trace the form of the curve from its polar equation.*

In (12) let $\theta = 0$. Then $r = a(e - 1) = F'A'$. As θ increases from 0 past $90°$, r increases and becomes infinite when

$$1 + e \cos \theta = 0$$

or when $$\cos \theta = -\frac{1}{e}.$$

Thus, while θ increases from 0 to the angle whose cosine is $-\frac{1}{e}$, that portion of the curve is traced out which begins at

A' and passes through P to an indefinite distance from the vertex. As θ increases from the angle whose cosine is $-\frac{1}{e}$ to $180°$, r is *negative* and decreases; hence the portion $P'A$ in the lower right-hand quadrant is traced out. When $\theta = 180°$, $r = -a(e+1) = F''A$. As θ increases from $180°$ to the angle whose cosine is $-\frac{1}{e}$ in the third quadrant, r is negative and increases numerically, and becomes indefinitely great when $\cos\theta = -\frac{1}{e}$. Thus the portion AP'' is traced out. As θ increases from that angle in the third quadrant whose cosine is $-\frac{1}{e}$ to $360°$, r again becomes positive, is at first indefinitely great and then diminishes until $\theta = 360°$, when $r = a(e-1) = F'A'$, as it should. Thus the portion $P'''A'$ in the lower left-hand quadrant is traced out.

EXERCISES.

1. Prove the following propositions:

I. The distance of each focus from the centre is *ae*.

II. The distance of each focus from the nearer vertex is $a(e-1)$, and from the farther vertex $a(e+1)$.

III. The distance between the vertices of the hyperbola and of its conjugate is equal to that between the centre and the foci.

IV. If we put e' for the eccentricity of the conjugate hyperbola, we shall have

$$e^2 + e'^2 = e^2e'^2.$$

V. The eccentricity of an equilateral hyperbola and of its conjugate are each $\sqrt{2}$.

2. Find the semi-axes and eccentricity of the hyperbola $16x^2 - 9y^2 = 144$. *Ans.* $a = 3$; $b = 4$; $e = \frac{5}{3}$.

3. Find the eccentricity and semi-parameter of the hyperbola $36x^2 - 25y^2 = 900$. *Ans.* $e = \frac{\sqrt{61}}{5}$; $p = 7.2$.

4. What is the equation of the hyperbola when the distance between the foci is 6 and the difference of the focal radii of any point of the curve is 4?

Ans. $5x^2 - 4y^2 = 20$.

5. The distance from the focus of an hyperbola to the more remote vertex is 4 and the eccentricity is $\frac{5}{3}$. Find the equation of the curve and its latus rectum.

Ans. $\frac{4}{9}x^2 - \frac{1}{4}y^2 = 1$; latus rectum $= \frac{16}{3}$.

6. What is the equation of the hyperbola whose transverse axis is 10 and whose vertex bisects the distance between the centre and the focus? *Ans.* $3x^2 - y^2 = 75$.

7. The equation of an hyberbola is $x^2 - 4y^2 = 12$. Find the equation of the conjugate hyperbola and its eccentricity.

Ans. $x^2 - 4y^2 = -12$; $e = \sqrt{5}$.

8. If e and e' denote the eccentricity of an hyperbola and its conjugate, show that

$$\frac{e}{e'} = \frac{b}{a}.$$

9. Find the equation of the hyperbola when the left-hand focus is the origin.

Ans. $\frac{y^2}{b^2} - \frac{x^2}{a^2} + \frac{2e}{a}x = e^2 - 1$.

10. Show that by multiplying every ordinate y of an ellipse referred to its centre and axes by the imaginary unit $\sqrt{-1}$, it will be changed into an hyperbola having the same axes.

11. A line parallel to the transverse axis is drawn so as to intersect both an hyperbola and its conjugate. Show that the segments contained between the two hyperbolas diminish indefinitely as the line recedes indefinitely. Also, that the rectangle contained by one of those segments, and by the sum of the two segments, one of which is cut out of the line by each hyperbola, is equal to the square upon the transverse axis.

Diameters of the Hyperbola.

158. *Def.* A **diameter** of an hyperbola is any line passing through the centre. The length of a diameter is the distance between the points in which it meets the curve.

THEOREM II. *Every diameter of an hyperbola or of its conjugate is bisected by the centre.*

Proof. Let the equation of any line through the centre of an hyperbola be

$$y = mx. \qquad (a)$$

The equation of the hyperbola is

$$b^2x^2 - a^2y^2 = a^2b^2, \qquad (b)$$

and the equation of the conjugate hyperbola,

$$b^2x^2 - a^2y^2 = -a^2b^2. \qquad (c)$$

Solving (a) and (b) for x and y, we have

$$\left.\begin{aligned} x &= \pm\frac{ab}{\sqrt{b^2 - a^2m^2}} \\ \text{and} \qquad y &= \pm\frac{mab}{\sqrt{b^2 - a^2m^2}}. \end{aligned}\right\} \qquad (d)$$

And solving (a) and (c) for x and y, we have

$$\left.\begin{aligned} x &= \pm\frac{ab}{\sqrt{a^2m^2 - b^2}} \\ \text{and} \qquad y &= \pm\frac{mab}{\sqrt{a^2m^2 - b^2}}. \end{aligned}\right\} \qquad (e)$$

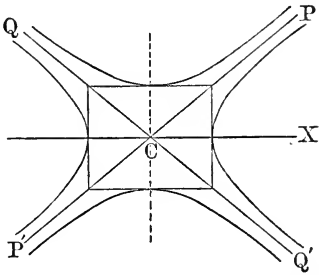
From (d) and (e) we see that the points of intersection of the line $y = mx$ with the hyperbola and its conjugate are at equal distances on each side of the origin. Q. E. D.

When $b^2 > a^2m^2$ or $m < \pm\frac{b}{a}$, the values of x and y in (d) are *real*, which shows that the line (a) intersects the given hyperbola at finite distances from the centre; while in (e) the values of x and y are *imaginary*, which shows that the line (a) does not then intersect the conjugate hyperbola.

If $b^2 < a^2m^2$ or $m > \pm \frac{b}{a}$, the values of x and y in (d) are *imaginary*, while in (e) they are *real*, showing that the line does not then meet the given hyperbola, but meets the conjugate at finite distances from the centre.

159. *Asymptotes.* If $b^2 = a^2m^2$ or $m = \pm \frac{b}{a}$, the values of x and y in both (4) and (5) become infinite. Hence the diameter whose slope to the transverse axis is either $+\frac{b}{a}$ or $-\frac{b}{a}$ meets the hyperbola or its conjugate only at infinity.

Def. That diameter of an hyperbola which meets the hyperbola and its conjugate at infinity is called an **asymptote** of the hyperbola.



Cor. 1. The equation of the asymptote CP is

$$y = \frac{b}{a}x, \quad \text{or} \quad \frac{x}{a} - \frac{y}{b} = 0; \quad (13)$$

and of the asymptote CQ,

$$x = -\frac{b}{a}x, \quad \text{or} \quad \frac{x}{a} + \frac{y}{b} = 0. \quad (14)$$

Cor. 2. Equations (13) and (14) are the equations of the diagonals of the rectangle formed by the axes of the curve. Hence:

THEOREM III. *The asymptotes coincide with the diagonals of the rectangle contained by the transverse and conjugate axes.*

160. THEOREM IV. *The locus of the centres of parallel chords of an hyperbola is a diameter.*

The demonstration of this theorem is similar in every respect to that of Theorem III. of the Ellipse. Substituting $-b^2$ for b^2 in §123 of the Ellipse, we have, omitting the accents on the variables,

$$-b^2x + a^2my = 0,$$

or

$$y = \frac{b^2}{a^2m}x, \quad (15)$$

as the equation of the locus of the centres of parallel chords. This is the equation of a straight line through the centre, and is therefore a diameter of the curve. By giving m suitable values, (15) may be made to represent any line through the centre. Hence

Every diameter bisects some system of parallel chords.

If m' be the slope to the transverse axis of any diameter which bisects a system of parallel chords whose slope is m, then the equation of the diameter is

$$y = m'x.$$

But, by (15),

$$y = \frac{b^2}{a^2m}x$$

is also the equation of the diameter;

therefore $$m' = \frac{b^2}{a^2m},$$

or $$mm' = \frac{b^2}{a^2}, \tag{16}$$

which is the relation which must hold between the slope of any system of parallel chords and the slope of the diameter which bisects these chords. Hence:

THEOREM V. *If one diameter bisects chords parallel to a second diameter, the latter will bisect all chords parallel to the former.*

161. *Conjugate Diameters.*

Def. Two diameters are said to be **conjugate to each other** when each bisects all the chords parallel to the other.

The equation of condition for *conjugate* diameters is, by (16),

$$mm' = \frac{b^2}{a^2},$$

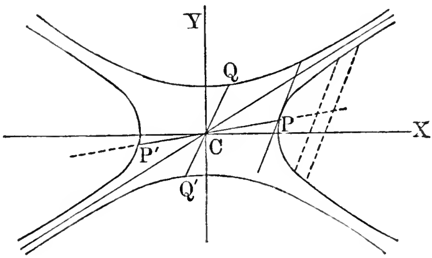
where m and m' denote their respective slopes to the transverse axis. Since the second member of this equation is positive, m and m' must have the same signs; that is, they must be both *positive* or both *negative.* Hence the angles which *conjugate diameters* make with the transverse axis must be both *acute* or both *obtuse.*

If $\qquad m < \frac{b}{a}, \qquad m' > \frac{b}{a};$

$$m > -\frac{b}{a}, \qquad m' < -\frac{b}{a};$$

and if $\qquad m = \pm\frac{b}{a}, \qquad m' = \pm\frac{b}{a}.$

Whence it follows that the conjugate diameters of an hyperbola lie on the same side of the *conjugate axis,* but on opposite sides of an asymptote; so that if one of two conjugates, as PP',



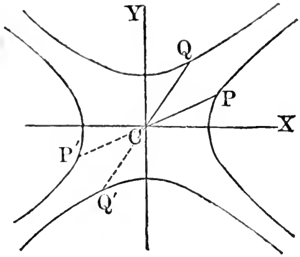
meets the hyperbola, the other, QQ', will meet the conjugate hyperbola. PP' produced bisects all chords parallel to QQ' in either branch of the hyperbola, and QQ', produced if necessary, bisects all chords drawn *between* the two branches of the curve and parallel to PP'.

Conversely, QQ' produced bisects all chords of either branch of the conjugate hyperbola parallel to PP', and PP' produced bisects all chords parallel to QQ' *between* the two branches of the conjugate hyperbola.

Cor. Since the chords of a set become indefinitely short near the extremity of the bisecting diameter, they will coincide in *direction* with the tangent at that point. Hence:

THEOREM VI. *The tangent to an hyperbola at the end of a diameter is parallel to the conjugate diameter.*

162. PROBLEM. *Given the co-ordinates of the extremity of one diameter, to find those of either extremity of the conjugate diameter.*



Let PP' and QQ' be any two conjugate diameters, and (x', y') the co-ordinates of P.

The equation of CP is

$$y = \frac{y'}{x'}x, \quad \text{since } m = \frac{y'}{x'}, \qquad (a)$$

and the equation of QQ' is

$$y = \frac{b^2}{a^2m}x$$

or

$$y = \frac{b^2x'}{a^2y'}x; \qquad (b)$$

and the equation of the conjugate hyperbola is

$$b^2x^2 - a^2y^2 = -a^2b^2. \qquad (c)$$

Solving (b) and (c), we have, since $b^2x'^2 - a^2y'^2 = a^2b^2$,

$$x = \pm\frac{a}{b}y'$$

and

$$y = \pm\frac{b}{a}x'.$$

163. THEOREM VII. *The difference of the squares of two conjugate semi-diameters is constant and equal to the difference of the squares of the semi-axes.*

Proof. Let (x', y') be the co-ordinates of P (last figure). Then the co-ordinates of Q will be

$$\left(\frac{a}{b}y', \frac{b}{a}x'\right).$$

If the semi-conjugates be denoted by a' and b', we have

$$CP^2 - CQ^2 = (x'^2 + y'^2) - \left(\frac{a^2}{b^2}y'^2 + \frac{b^2}{a^2}x'^2\right)$$
$$= \frac{b^2x'^2 - a^2y'^2}{b^2} - \frac{b^2x'^2 - a^2y'^2}{a^2},$$

or

$$a'^2 - b'^2 = a^2 - b^2 \qquad (17)$$
$$= \text{a constant.}$$

164. PROBLEM. *To express the angle between two conjugate diameters in terms of their lengths.*

Let θ and θ' denote the angles which the conjugate semi-diameters CP, CQ make with the transverse axis, and φ the angle PCQ between them.

Then $$\varphi = \theta' - \theta$$

and $$\sin\varphi = \sin\theta'\cos\theta - \sin\theta\cos\theta'. \qquad (a)$$

Now if (x', y') be the co-ordinates of P, those of Q will be

$$\left(\frac{a}{b}y',\ \frac{b}{a}x'\right);$$

therefore

$$\sin\theta = \frac{y'}{a'}; \qquad \cos\theta = \frac{x'}{a'};$$

$$\sin\theta' = \frac{bx'}{ab'}; \qquad \cos\theta' = \frac{ay'}{bb'}.$$

Substituting in (a), we get

$$\sin\varphi = \frac{b^2x'^2 - a^2y'^2}{aba'b'} = \frac{ab}{a'b'}, \qquad (18)$$

the required expression.

Cor. From (17) we have

$$a'^2 = b'^2 + \text{a constant};$$

therefore a' and b' increase together or decrease together. Hence, when each tends to coincide with the asymptote, the product $a'b'$ tends towards infinity, and $\sin\varphi$ tends towards 0; therefore the angle between two conjugates diminishes without limit. When the conjugates coincide with the asymptotes, each becomes infinite.

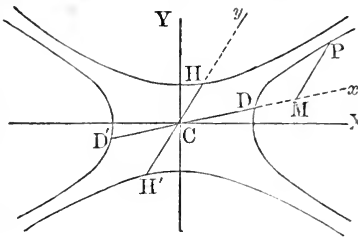
165. THEOREM VIII. *The area of the parallelogram whose sides touch an hyperbola at the ends of any pair of conjugate diameters is constant and equal to the rectangle formed by the axes of the curve.*

Proof. From (18) we have

$$\begin{aligned} 4a'b'\sin\varphi &= 4ab \\ &= \text{a constant}, \end{aligned} \qquad (19)$$

which proves the proposition.

166. Problem. *To find the equation of the hyperbola referred to a pair of conjugate diameters as axes.*



Let DD', HH' be any pair of conjugate diameters. Take DD' for the new axis of X, and HH' for the new axis of Y, and let the angle $XCD = \alpha$ and $XCH = \beta$. We may now transform

$$b^2x^2 - a^2y^2 = a^2b^2$$

from rectangular to oblique axes by the process of § 129, or we may simply change b^2 into $-b^2$ in equation (12) of that section. Thus we get

$$(a^2 \sin^2\alpha - b^2 \cos^2\alpha)x'^2 + (a^2 \sin^2\beta - b^2 \cos^2\beta)y'^2 = -a^2b^2, \quad (20)$$

which is the equation required.

By putting x' and y' each equal to zero, we get the intercepts on the axes or the lengths of the semi-conjugates. Thus, when $y'^2 = 0$,

$$x'^2 = \frac{-a^2b^2}{a^2 \sin^2\alpha - b^2 \cos^2\alpha} = CD^2 = a'^2; \quad (a)$$

and when $x'^2 = 0$, we have

$$y'^2 = -\frac{a^2b^2}{a^2 \sin^2\beta - b^2 \cos^2\beta} = -CH^2 = -b'^2. \quad (b)$$

Because the new axis of X meets the given hyperbola, the new axis of Y will not meet the curve, but will meet the conjugate hyperbola. Therefore $\dfrac{-a^2b^2}{a^2 \sin^2\beta - b^2 \cos^2\beta}$ is a negative quantity.

From (a) and (b) we get

$$a^2 \sin^2\alpha - b^2 \cos^2\alpha = -\frac{a^2b^2}{a'^2}$$

and

$$a^2 \sin^2\beta - b^2 \cos^2\beta = \frac{a^2b^2}{b'^2}.$$

Substituting in (20) and dividing by $-a^2b^2$, we have, omitting the accents from the variables,

$$b'^2x^2 - a'^2y^2 = a'^2b'^2,$$

or
$$\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = 1. \qquad (21)$$

Also, the equation of the conjugate hyperbola referred to the same axes is

$$\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = -1.$$

Tangent and Normal to an Hyperbola.

167. PROBLEM. *To find the equation of the tangent to an hyperbola.*

In order to obtain the equation of the tangent, we have only to repeat the process of §§ 134, 135, changing b^2 into $-b^2$. Thus we get

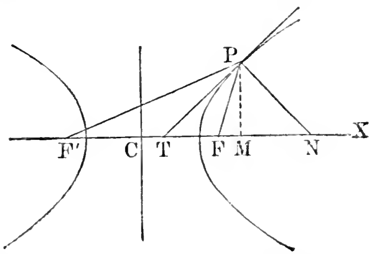
$$a^2y'y - b^2x'x = -a^2b^2,$$

or
$$\frac{xx'}{a^2} - \frac{yy'}{b^2} = 1, \qquad (22)$$

and also
$$y = mx \pm \sqrt{a^2m^2 - b^2}, \qquad (23)$$

where m is the slope of the tangent to the transverse axis.

Intercept of the Tangent on the Axis of X.



In (22) make $y = 0$. Then

$$x = \frac{a^2}{x'} = CT, \qquad (24)$$

from which we see that x and x' must always have the same sign; and since x is always positive in the right branch of the curve, the tangent to that branch always intersects the axis of X to the right of the centre.

168. *Subtangent.* For the length of the subtangent we have, from the figure,

$$\text{Subtangent} = MT$$
$$= x' - \frac{a^2}{x'} = \frac{x'^2 - a^2}{x'}. \qquad (25)$$

169. THEOREM IX. *The tangent to an hyperbola at any point bisects the angle formed by the focal radii of that point.*

Proof. Since $F'C = FC = ae$ and $CT = \frac{a^2}{x'}$, we have

$$F'T = ae + \frac{a^2}{x'} = \frac{a}{x'}(ex' + a)$$

and
$$FT = ae - \frac{a^2}{x'} = \frac{a}{x'}(ex' - a);$$

whence
$$\frac{F'T}{FT} = \frac{ex' + a}{ex - a}$$

$$= \frac{F'P}{FP}. \qquad (\S\,154)$$

Therefore, since the base of the triangle $F'FP$ is divided proportionally to its sides (Geom.), the tangent PT bisects the angle FPF'.

170. *Tangent through a Given Point.*

Let (h, k) be the co-ordinates of the given point, and (x', y') the co-ordinates of the point of contact. The equation of the tangent is

$$b^2x'x - a^2y'y = a^2b^2;$$

but since the tangent must pass through (h, k) and (x', y'), we have

$$b^2hx' - a^2ky' = a^2b^2, \qquad (a)$$

and also
$$b^2x'^2 - a^2y'^2 = a^2b^2. \qquad (b)$$

Eliminating y' from these equations, we have

$$(a^2k^2 - b^2h^2)x'^2 + 2a^2b^2hx' - a^4(b^2 + k^2) = 0;$$

whence

$$x' = \frac{a^2b^2h \mp a^2k\sqrt{a^2k^2 - b^2h^2 + a^2b^2}}{b^2h^2 - a^2k^2}. \qquad (26)$$

Since x' has two values, two tangents to an hyperbola can be drawn through a given fixed point. The tangents will be *real, coincident* or *imaginary* according as

$$a^2k^2 - b^2h^2 + a^2b^2 >, \quad = \quad \text{or} \quad < \quad 0;$$

that is, according as the given point is *within, on* or *outside* the curve.

171. PROBLEM. *To find the criterion that the two tangents from a given point shall touch the same branch of the hyperbola.*

If the tangents belong to the *same* branch of the curve, the abscissæ of the points of contact x' will have *like* signs; but if they belong to opposite branches, *unlike* signs. Now, in order that the values of x' in (26) may have like signs, we must have numerically

$$a^2b^2h > a^2k\sqrt{a^2k^2 - b^2h^2 + a^2b^2};$$

whence, by reduction,

$$k < \frac{b}{a}h.$$

But
$$y = \frac{b}{a}x$$

is the equation of the asymptote; and if we take on the asymptote a point whose abscissa x is equal to the abscissa h of the point from which two tangents may be drawn, we shall have $k > y$; that is, the ordinate of the point from which two tangents can be drawn to the same branch of an hyperbola must be *less* than the corresponding ordinate of the asymptote. Hence the point from which two tangents can be drawn to the *same* branch of an hyperbola must lie in the space between the asymptotes and the *adjacent* branch of the curve, which is the required criterion.

Hence, also, if the point lie without this space, the two tangents will touch different branches of the curve.

172. PROBLEM. *To find the locus of the point from which two tangents to an hyperbola make a right angle with each other.*

The solution is similar to the corresponding problem in the Ellipse. We will therefore simply change b^2 to $-b^2$ in the process of § 138, and we get

$$x^2 + y^2 = a^2 - b^2$$

for the required locus, which is a circle having the same centre as that of the hyperbola and whose radius is $\sqrt{a^2 - b^2}$.

Cor. Two tangents at right angles to each other cannot be drawn to an hyperbola when $b > a$.

173. PROBLEM. *To find the locus of the intersection of the tangent with the perpendicular on it from the focus.*

The solution is the same as that of the corresponding problem in the case of the ellipse.

The equation of the required locus is found to be

$$x^2 + y^2 = a^2,$$

which is a circle described on the transverse axis as a diameter.

174. PROBLEM. *To find the length of the perpendicular from either focus upon the tangent to an hyperbola.*

If (x', y') be the co-ordinates of the point of tangency, and p, p' the perpendiculars from the foci F and F' respectively, we find, in the same manner as in the Ellipse,

$$\left.\begin{aligned} p &= \frac{ab^2(ex' - a)}{\sqrt{b^4x'^2 + a^4y'^2}}; \\ p' &= \frac{ab^2(ex' + a)}{\sqrt{b^4x'^2 + a^4y'^2}}; \end{aligned}\right\} \tag{27}$$

whence we get, by reduction,

$$pp' = b^2 \tag{28}$$

and

$$\frac{p}{p'} = \frac{ex - a}{ex + a} = \frac{r}{r'}, \tag{29}$$

where r and r' denote the focal radii of the point of contact.

From the last two equations we readily find

$$p^2 = \frac{r}{r'}b^2, \qquad p'^2 = \frac{r'}{r}b^2$$

and

$$p^2 = \frac{rb^2}{2a + r}. \tag{30}$$

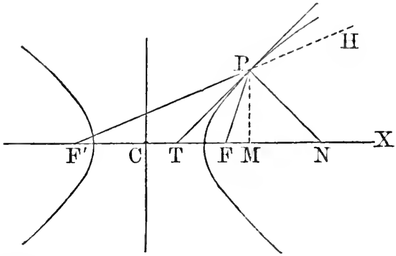
175. *Normal to an Hyperbola.*

PROBLEM. *To find the equation of the normal to an hyperbola.*

The equation of the normal is found by changing b^2 into $-b^2$ in the process of § 141. Thus, if (x', y') be the co-

ordinates of any point P on the hyperbola, the equation of the normal PN is

$$y - y' = -\frac{a^2}{b^2}\frac{y'}{x'}\left(x - x'\right). \tag{31}$$



The Subnormal. Putting $y = 0$ in (31), we find

$$x = \frac{a^2 + b^2}{a^2}x' = e^2x' = CN.$$

Hence, subnormal $= MN = CN - CM = (e^2 - 1)x'$.

176. THEOREM X. *The normal at any point of an hyperbola bisects the external angle contained by the focal radii of that point.*

Since the angles FPF' and FPH are supplementary and TP bisects FPF', therefore PN, which is perpendicular to TP, must bisect the external angle FPH.

Cor. Comparing this result with that of § 143, we see that *if an ellipse and an hyperbola have the same foci, the curves will intersect at right angles.*

For at the point of intersection the tangent of one will be the normal of the other, and *vice versa.*

REMARK. The student should note the relations between the different theorems and formulæ relating to the ellipse and the corresponding ones relating to the hyperbola. Where the formula of the one class contains the symbol b^2, it may be applied immediately to the other by changing the sign of b^2, which will be the result of substituting $b\sqrt{-1}$ for b. Where only the first power of b enters, the theorems of one class involving *real* quantities will be *imaginary* when transferred to the other class. Thus we have imaginary asymptotes to the ellipse. The apparent exceptions arise from our substituting a real for an imaginary conjugate axis in the hyperbola and thus referring several expressions which would have been imaginary to the conjugate hyperbola, which, it must be remembered, is not a part of the curve at all.

Poles and Polars.

177. PROBLEM. *To find the equation of the chord of contact of two tangents from the same given point.*

Let (h, k) be the co-ordinates of the fixed point from which the two tangents that determine the chord are drawn. Then, by simply changing the sign of b^2 in § 144, the equation of the hyperbolic chord of contact will be

$$\frac{h'x}{a^2} - \frac{k'y}{b^2} = 1 \tag{32}$$

when referred to the axes of the curve,

or

$$\frac{h'x}{a'^2} - \frac{k'y}{b'^2} = 1 \tag{33}$$

when referred to any pair of conjugate diameters.

178. *Locus of Intersection of Two Tangents whose chord of contact passes through a fixed point.*

Let (x', y') be the co-ordinates of any fixed point through which the chord of contact belonging to any two intersecting tangents is drawn. Then, by simply changing the sign of b^2 in the process of § 145, we shall have, for the equation of the required locus,

$$\frac{x'x}{a^2} - \frac{y'y}{b^2} = 1; \tag{34}$$

or, when referred to a pair of conjugate diameters,

$$\frac{x'x}{a'^2} - \frac{y'y}{b'^2} = 1, \tag{35}$$

which is the equation of a straight line, the **polar** of the point (x', y').

Cor. The student may easily show, as in the case of the ellipse, that the *polar* of any point in respect to an hyperbola is parallel to the diameter conjugate to that which passes through the point.

179. *Polars of Special Points.*

Polar of the Centre. Proceeding in the same manner as in the ellipse, we find that the polar of the centre is at infinity.

The Polar of any Point on a Diameter A is a straight line parallel to the conjugate diameter, and cutting the diameter A at a distance from the centre equal to the square of the semi-diameter on which the point is taken divided by the distance of the point from the centre.

Polar of the Focus. Substituting $(\pm ae, 0)$ for $(x,' y')$ in the equation of the polar, we have

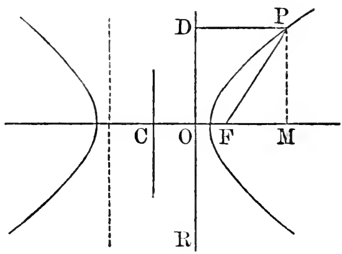
$$x = \pm \frac{a}{e}.$$

Hence *the polar of the focus of an hyperbola is the perpendicular which cuts the transverse axis at a distance $\frac{a}{e}$ from the centre on the same side as the focus.*

180. *Distance of any Point on the Curve from either Focal Polar.*

Let DR be the polar of the focus F. Then we have

$$\begin{aligned} OC &= \frac{a}{e}; \\ DP &= OM \\ &= CM - CO \\ &= x - \frac{a}{e} \\ &= \frac{ex - a}{e} \\ &= \frac{FP}{e}; \end{aligned}$$



therefore $\frac{FP}{DP} = e.$

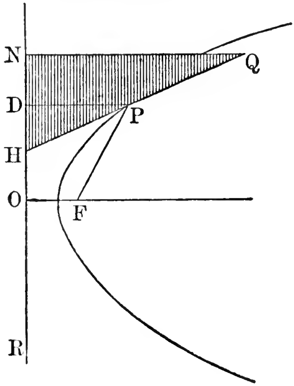
Whence:

THEOREM XI. *The focal distance of any point on an hyperbola is in a constant ratio to its distance from the polar of the focus.*

This ratio is greater than unity and equal to the eccentricity of the curve.

Def. The polar of the focus is called the **directrix** of the hyperbola.

The above property enables us to describe the curve by continuous motion, as follows: Take any fixed straight line NR and any fixed point F, and against the former fasten a ruler, and place another ruler, right-angled at N, so that its edge, NH, may move freely along NR. At F attach one end of a thread equal in length to the hypothenuse HQ of the ruler, and the other end to the extremity Q of the ruler. Then with a pencil-point P stretch the thread tightly against the edge HQ, while the ruler is moved along the other ruler, NR. The point P will describe an hyperbola, for in every position we shall have



$$PF = PH;$$

and therefore

$$\frac{PF}{PD} = \frac{PH}{PD}$$

$$= \frac{HQ}{NQ} = \text{a constant.}$$

181. *Cor.* From §§ 97, 148 and 180 it follows that we may define a conic section as the locus of a point which moves in such a way that its distance from a fixed point (the focus) is in a constant ratio to its distance from a fixed straight line (the directrix).

In the ellipse, the ratio $\dfrac{PF}{PD} < 1.$

In the parabola, the ratio $\dfrac{PF}{PD} = 1.$

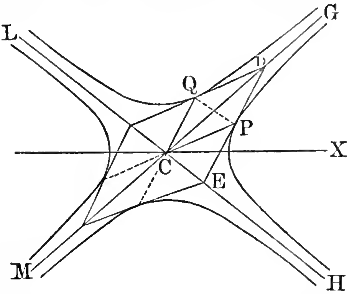
In the hyperbola, the ratio $\dfrac{PF}{PD} > 1.$

In all cases this ratio is the eccentricity of the curve.

In the case of the ellipse and hyperbola there is a directrix corresponding to each focus. In the case of the parabola the second focus and directrix are at infinity.

The Asymptotes.

182. We have already shown (§ 159) that the equations of the asymptotes when referred to rectangular co-ordinates are



$$\frac{x}{a} \pm \frac{y}{b} = 0. \qquad (a)$$

Now since the equation of the hyperbola referred to a pair of conjugate diameters as axes is of the same form as when referred to rectangular axes, we at once infer that equations (a) transformed to the same conjugate diameters become

$$\frac{x}{a'} \pm \frac{y}{b'} = 0;$$

that is, the equations of the asymptotes MG, LH when referred to any pair of conjugate diameters are respectively

$$\frac{x}{a'} - \frac{y}{b'} = 0 \qquad (b)$$

and
$$\frac{x}{a'} + \frac{y}{b'} = 0. \qquad (c)$$

Equation (b) is the equation of a line which passes through the centre or origin and the point $(+a', +b')$; that is, through C and D; and (c) is the equation of a line which passes through the origin and the point $(+a', -b')$ or C and E. Hence we conclude:

THEOREM XI. *The asymptotes coincide in direction with the diagonals of the parallelogram formed by any pair of conjugate diameters.*

183. *Angle between the Asymptotes.*

Let $GCX = \alpha$. Then $\tan\alpha = \dfrac{b}{a}$;

whence $\sin\alpha = \dfrac{b}{\sqrt{a^2+b^2}}$ and $\cos\alpha = \dfrac{a}{\sqrt{a^2+b^2}}$.

Now since $XCH = GCX$, $\sin GCH = \sin 2GCX$;

hence $$\sin GCH = 2 \sin GCX \cos GCX$$

$$= \frac{2ab}{a^2 + b^2}.$$

Cor. In the equilateral hyperbola, $a = b$;

hence $$\sin GCH = 1;$$

that is, the asymptotes of the equilateral hyperbola intersect at right angles. For this reason the equilateral hyperbola is sometimes called the *rectangular* hyperbola.

184. THEOREM XII. *The asymptotes of the hyperbola are its tangents at infinity.*

We prove this by showing that, as the point of tangency on an hyperbola recedes indefinitely, the tangent approaches the asymptote as its limit.

1. If, in equation (24) of § 167,

$$x = \frac{a^2}{x'},$$

we suppose x' to increase without limit; x, the abscissa of the point in which the tangent intersects the transverse axis, approaches zero as its limit. Hence the tangent at infinity passes through the centre of the hyperbola.

2. From the equation of the tangent,

$$b^2x'x - a^2y'y = a^2b^2,$$

it follows that its slope to the axis of X is $\frac{b^2x'}{a^2y'}$. We must now find the value of this slope when the point of tangency (x', y') recedes to infinity. Because this point remains on the hyperbola, we have

$$\frac{x'^2}{a^2} - \frac{y'^2}{b^2} = 1;$$

whence

$$\frac{x'}{y'} = \sqrt{\left(\frac{a^2}{b^2} + \frac{a^2}{y'^2}\right)}.$$

As y' recedes to infinity, $\frac{b^2}{y'^2}$ approaches zero as its limit; whence, at infinity, $\frac{x'}{y'} = \pm \frac{a}{b}$, and we have

$$\text{Slope of tangency at infinity} = \pm \frac{b}{a}.$$

Hence the tangents at infinity are a pair of lines whose equations are

$$y = \pm \frac{b}{a}x,$$

which lines are the asymptotes, by definition.

185. PROBLEM. *To find the equation of the hyperbola referred to its asymptotes as axes.*

Let the asymptote CH be the new axis of X, and the other, CG, the new axis of Y; (x, y) be the co-ordinates of any point on the curve referred to the old axes, and (x', y') the co-ordinates of the same point referred to the new axes.

The equation of the curve referred to the old axes is

$$b^2x^2 - a^2y^2 = a^2b^2, \tag{a}$$

which must be transformed to the new or oblique axes, the origin remaining the same.

The formulæ of transformation are

$$\left.\begin{aligned} x &= x' \cos \alpha + y' \cos \beta; \\ y &= x' \sin \alpha + y' \sin \beta; \end{aligned}\right\} \tag{b}$$

where α and β are the angles which the new axes make with the old axis of X; that is, $\alpha = XCH$ and $\beta = GCX$,

or $$\beta = -\alpha.$$

Therefore (b) becomes

$$x = (x' + y')\cos \alpha;$$
$$y = (x' - y')\sin \alpha;$$

which being substituted in (a) give, after obvious reductions,

$$(b^2 \cos^2\alpha - a^2 \sin^2\alpha)(x'^2 + y'^2) + 2(b^2 \cos^2\alpha + a^2 \sin^2\alpha)x'y' = a^2b^2.$$

But $$\tan \alpha = \frac{b}{a}, \quad \text{or} \quad \frac{\sin \alpha}{\cos \alpha} = \frac{b}{a};$$

whence $$b^2 \cos^2\alpha = a^2 \sin^2\alpha;$$

which being substituted in the preceding equation gives, after dividing by a^2,

$$4 \sin^2\alpha\, x'y' = b^2.$$

But $$\sin \alpha = \frac{b}{\sqrt{a^2 + b^2}}; \qquad (\S\ 183)$$

therefore $$4x'y' = a^2 + b^2,$$

or, omitting the accents on the variables, since the equation is perfectly general,

$$xy = \frac{a^2 + b^2}{4}, \qquad (36)$$

which is the required equation.

Cor. The equation of the *conjugate* hyperbola referred to the same axes is readily found to be

$$xy = -\frac{a^2 + b^2}{4}. \qquad (37)$$

186. PROBLEM. *To find the equation of the tangent to an hyperbola referred to the asymptotes as axes.*

Let (x', y') and (x'', y'') be the co-ordinates of any two points on the curve. The equation of the secant through these two points is

$$y - y' = \frac{y' - y''}{x' - x''}(x - x'). \qquad (a)$$

Since the points (x', y') and (x'', y'') are on the curve,

$$x'y' = \frac{a^2 + b^2}{4}$$

and $$x''y'' = \frac{a^2 + b^2}{4};$$

whence $$x'y' = x''y'', \qquad \text{or} \qquad y'' = \frac{x'y'}{x''},$$

which substituted in (a) gives, after reduction,

$$y - y' = -\frac{y'}{x''}(x - x').$$

Now at the limit, $x'' = x'$ and the secant becomes a tangent; hence the equation of the tangent at the point (x', y') is

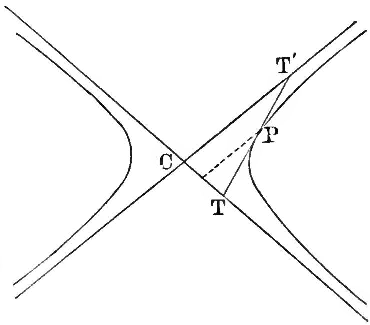
$$y - y' = -\frac{y'}{x'}(x - x');$$

whence $$x'y + xy' = 2x'y',$$

or $$\frac{x}{x'} + \frac{y}{y'} = 2, \tag{38}$$

which is the simplest form of the required equation.

Cor. 1. Making x and y successively equal to 0, we get the intercepts on the axes;



thus, $x = 2x' = CT$

and $y = 2y' = CT'$.

Hence the point of contact is the middle point of TT'; or, that portion of a tangent intercepted between the asymptotes is bisected at the point of contact.

Cor. 2. $$CT \times CT' = 4x'y' = a^2 + b^2;$$

or, the rectangle formed by the intercepts cut off by any tangent from the asymptotes is constant and equal to the sum of the squares of the semi-axes.

Cor. 3. The area of the triangle CTT' is

$$\begin{aligned} &= \tfrac{1}{2} CT \,.\, CT' \,.\, \sin TCT' \\ &= 2x'y' \times \frac{2ab}{a^2 + b^2} \\ &= \frac{a^2 + b^2}{2} \cdot \frac{2ab}{a^2 + b^2} \\ &= ab, \text{ a constant}; \end{aligned}$$

or, the area of the triangle formed by any tangent and the asymptotes is constant and equal to the rectangle of the semi-axes.

EXERCISES.

1. Find the equation of that hyperbola whose transverse axis is 8 and which passes through the point (10, 25).

$$\textit{Ans.}\ \frac{x^2}{16} - \frac{21y^2}{2500} = 1.$$

2. What condition must the eccentricity of an hyperbola fulfil in order that the abscissa of some point upon it shall be equal to the ordinate? *Ans.* $e < \sqrt{2}$.

3. Express the distance from the centre of an hyperbola to the end of its parameter in terms of the semi-transverse axis and eccentricity.

4. Show that each ordinate of an equilateral hyperbola is a mean proportional between the sum and difference of the abscissa and semi-transverse axis.

5. Write the equation of a focal chord cutting an hyperbola at the point (x', y').

6. Find that point upon the conjugate axis from which the two tangents to an hyperbola form a right angle with each other.

7. Where do the tangents drawn from a vertex of the conjugate hyperbola touch the hyperbola, and what are the equations of these tangents? Show that they are bisected by the transverse axis.

8. What must be the eccentricity in order that the tangent at the end of the parameter may pass through the vertex of the conjugate hyperbola?

9. Find those tangents to an hyperbola which make an angle of 60° with the transverse axis.

10. What must be the eccentricity of an hyperbola that the subnormal may always be equal to the abscissa of the point from which the normal is drawn?

11. Find the equation of the hyperbola when the origin is transferred to one of the vertices, while the axes of coordinates remain parallel to the principal axes.

12. Express the product of the segments into which a

focal chord is divided by the focus in terms of the angle which the chord forms with the major axis.

Ans. $\dfrac{p^2}{1 - e^2\cos^2\theta}$.

13. Show that the sum of the reciprocals of the two segments of a focal chord is equal to four times the reciprocal of the parameter.

14. The line $x = 3y$ is a diameter of the hyperbola $25x^2 - 16y^2 = 400$. Find the equation of the conjugate diameter.

15. For what point of an hyperbola are the subtangent and subnormal equal to each other?

16. Express the length of the tangent at the point (x', y').

17. Find the condition that the line $\dfrac{x}{b} + \dfrac{y}{a} = 1$ shall touch the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$. *Ans.* $e^4 - e^2 = 1$.

18. A perpendicular is drawn from the focus of an hyperbola to an asymptote. Show that its foot is at distances a and b from the centre and focus respectively.

19. Show that the linear equation of the right-hand branch of the hyperbola when a focus is the origin is

$$r = ex \mp a(1 - e^2).$$

20. Each ordinate of an hyperbola is produced until it is equal to the focal radius of the point to which it belongs. Find the locus of its extremity.

21. Find the equation of the tangent at the extremity of the latus rectum.

22. Show that the intercepts cut off from the normal by the axes are in the ratio of $a^2 : b^2$.

23. In an hyperbola, $3a = 2c$. Find the eccentricity and the angle between the asymptotes.

Ans. $e = \dfrac{3}{2}$; $\sin^{-1}\dfrac{4}{9}\sqrt{5}$.

24. Show that the angle between the asymptotes of an hyperbola is

$$2\sec^{-1} e.$$

25. From any point on an hyperbola perpendiculars are drawn to the asymptotes. Show that their product is constant and equal to $\frac{a^2b^2}{a^2 + b^2}$.

26. From any point in one of the branches of the conjugate hyperbola tangents are drawn to an hyperbola. Show that the chord of contact touches the other branch of the conjugate hyperbola.

27. Show that the polar equations of the right-hand branch of an hyperbola referred to the foci are

$$r = \frac{a(e^2 - 1)}{1 - e\cos\theta} \qquad \text{and} \qquad r = \frac{a(1 - e^2)}{1 - e\cos\theta}.$$

28. Show that the polar equation of the hyperbola when the centre is the pole is

$$r^2 = \frac{b^2}{e^2\cos^2\theta - 1}.$$

29. Show that the length of any focal chord of an hyperbola is $\frac{2}{a} \cdot \frac{b^2}{e^2\cos^2\theta - 1}$, where θ is the inclination of the chord to the transverse axis.

30. In the figure of § 182, show that the diagonal PQ is parallel to the asymptote.

31. In an equilateral hyperbola, if φ is the inclination of a diameter passing through any point P, and φ' the inclination of the polar of P, show that

$$\tan\varphi \tan\varphi' = 1.$$

32. Through the point (5, 3) is to be drawn a chord to the hyperbola $25x^2 - 16y^2 = 400$ which shall be bisected by the point. Find the equation of the chord.

33. In an hyperbola is to be inscribed (or escribed) an equilateral triangle, one of whose vertices shall be at the right-hand vertex of the curve. Find the sides of the triangle, and find the eccentricity when they are infinite.

34. Express the tangent of the angle between the two focal radii drawn to the point (x', y') of an hyperbola, and

thence find those points of the curve from which these radii subtend a right angle.

Ans., in part. $\tan\varphi = \dfrac{2aey'}{x'^2 + y'^2 - a^2e^2}$.

35. For what point on an equilateral hyperbola is the product of the focal radii equal to $3b^2$?

36. From the foot of any ordinate of an hyperbola a tangent is drawn to that circle described upon the major axis as a diameter. Show that the ratio of the ordinate to the tangent is a constant and equal to $e^2 - 1$.

37. Find the lengths which the directrix of an hyperbola cuts off from the asymptotes, and the length of that segment of the directrix contained between the asymptotes.

Ans. a and $b \div e$.

38. Find the polar of the vertex of the conjugate hyperbola relatively to the principal hyperbola.

39. From any point of an hyperbola is drawn a parallel to the asymptote, terminating at the directrix. Find the ratio of the length of this parallel to the focal radius of the point, and show that it is a constant.

40. Show (1) that the sides of the quadrilateral whose vertices are at the termini of any pair of conjugate diameters are equally inclined to the principal axes; (2) that all such quadrilaterals in the same hyperbola have their corresponding sides parallel and are equal in area.

41. Find that point of an hyperbola for which the tangent is double the normal.

42. At what angle does the hyperbola $x^2 - y^2 = a^2$ intersect the circle $x^2 + y^2 = 9a^2$?

43. A line drawn perpendicular to the transverse axis of an hyperbola meets the curve and its conjugate in P and Q respectively. Find the loci of the intersection of the normals, and of the tangents, at P and Q.

Ans. The transverse axis; $\dfrac{y^2}{b^2} - \dfrac{x^2}{a^2} = 4\dfrac{b^4x^2}{a^2y^4}$.

44. The two sides of a constant angle slide along a parabola. Find the locus of the vertex of the angle, and compare the cases of two loci whose angles are supplementary.

The Equation of the Conic Section ~~obtained from Boscovich's definition~~ is

$$(x-x')^2+(y-y')^2 = e^2(x\cos\alpha+y\sin\alpha-p)$$

where $(x'y')$ is the focus and $x\cos\alpha + etc = 0$ is the equation of the directrix.

Choosing the directrix and a line perp. to it thro' the focus as axes and making distance from focus to directrix $= d$, the equation is

$$(x-d)^2+y^2 = e^2x^2 \qquad (1)$$

The curve cuts the axis of x at the point

$$x = \frac{d}{1\pm e}$$

Transfer the origin to this point. Then the equation is

$$(1-e^2)x^2+y^2 = 2ed\,\frac{e\pm1}{1\pm e}\,x \qquad (2)$$

If we choose the left-hand vertex of the ellipse and the right-hand vertex of the hyperbola as origin, the equation is

$$(1-e^2)x^2+y^2 = 2edx.$$

This gives also the equation of the parabola referred to the vertex as origin, e being $= 1$.

This then is the equation of the several conics. The criteria of classification are

$e=0$	Circle; d is here infinite
$e=1$	Parabola
$e<1$	Ellipse
$e>1$	Hyperbola.

The parameter is in each case $2ed$

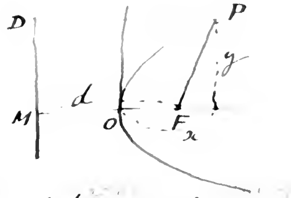
The equation of the conic may therefore be written

$$(1-e^2)x^2+y^2 = 2px \qquad (3)$$

where p is the semi-parameter.

For the circle $ed = r = p$. To show that ed is finite when $e=0$, $d=\infty$

$$e = \frac{FO}{OM} = \frac{FP}{PD} = \frac{FP}{d+x-OF}$$



$\therefore$ $ed + e(x-OF) = FP$

or $ed = FP$, when $e=0$

unless $x=\infty$; but e and d are constant and we may take x finite;

$\therefore$ FP or ed is finite and constant and P describes a circle whose centre is F.

If now e be equal to zero, we have

$$x^2+y^2 = 2px,$$

the equation of the circle.

For a pair of straight lines the parameter is zero and $e>1$.

The equation

$$(1-e^2)x^2+y^2 = 2px \qquad (3)$$

represents therefore all classes of conics.

For the circle $e=0$, $p=$ radius

" " parabola $e=1$, $p=$ semi-parameter

" " Ellipse $e<1$, $p=$ " "

" " hyperbola $e>1$, $p=$ " "

" " str. lines $e>1$, $p=0$.

The equation of the tangent at $x'y'$ is

$$(1-e^2)xx'+yy' = p(x+x')$$

Distances from origin to foci are

$$\frac{de}{1+e} \text{ and } \frac{de}{1-e} \text{ or } \frac{p}{1\pm e}$$

$\times \quad \frac{p}{a} = \frac{b^2}{a^2} = 1 - e^2$

With this substitution Equation (3) becomes

$$\frac{p}{a}x^2 + y^2 = 2px$$

or also

$$b^2x^2 + a^2y^2 = 2a^2px$$

$$\dagger \quad r + r' = f + f' = 2a$$

also

$$r = a(1-e) + ex$$

$$r' = a(1+e) - ex$$

Completing the square for x in equation (3), the latter becomes

$$\frac{\left(x - \frac{p}{1-e^2}\right)^2}{\left(\frac{p}{1-e^2}\right)^2} + \frac{y^2}{\left(\frac{p}{\sqrt{1-e^2}}\right)^2} = 1. \qquad (4)$$

Writing x for $x - \frac{p}{1-e^2}$, a for $\frac{p}{1-e^2}$ and b for $\frac{p}{\sqrt{1-e^2}}$ we have

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (4)$$

From $a = \frac{p}{1-e^2}$ and $b = \frac{p}{\sqrt{1-e^2}}$ we obtain

$$a - \frac{b^2}{p} = 0 \quad \text{or,} \quad p = \frac{b^2}{a} \qquad \times$$

If the origin be transferred to the focus $\left(\frac{d-e}{1+e}, 0\right)$, Equation (3) becomes

$$(1-e^2)x^2 + y^2 = 2pex + p^2 \qquad (5)$$

The focal radii to the point $(x\,y)$ are

$$r = f + ex \quad \text{and} \quad r' = f' - ex \qquad \dagger$$

where $f = \frac{p}{1+e}$ and $f' = \frac{p}{1-e}$

viz. the distances from the origin to the foci, the axis of reference being the major axis and the tangent at the vertex

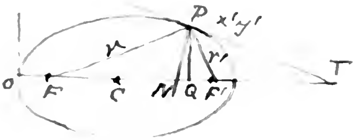
---

Subtangent.

The intercept of the tangent on X is

$x = \frac{ax'}{x'-a}$ x' being abscissa of pt. of contact

and subtangent is $\frac{x'(2a-x')}{x'-a}$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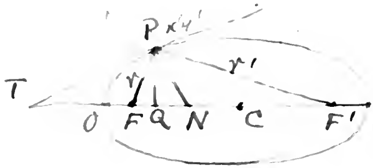
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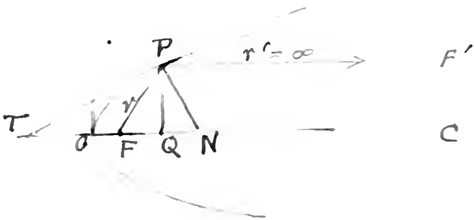
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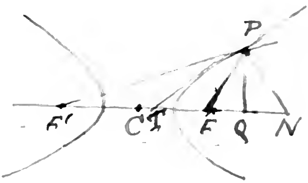
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Sub-normal to point $x'y'$

Sub-normal $= (1-e^2)(x'-a)$

$= x'(1-e^2) - p = MQ$

Dist. fr. centre to normal $= e^2(x'-a) = CN$

Dist. to centre fr. origin is $a = \dfrac{p}{1-e^2}$

" fr. focus F to centre is

$$FC = \frac{p}{1-e^2} - \frac{p}{1+e} = \frac{p - p(1-e)}{1-e^2} = \frac{pe}{1-e^2} = ae$$

$$FN = FC + CN = ae + e^2(x'-a)$$
$$= ae(1-e) + e^2x'$$
$$= e\left\{\frac{p}{1+e} + ex'\right\} = er$$

$$F'N = F'C + CN = -ae + e^2(x'-a)$$
$$= -ae(1+e) + e^2x'$$
$$= e\left\{-\frac{p}{1-e} + ex'\right\} = er'$$

$$\frac{r}{\sin FNP} = \frac{FN}{\sin FPN}; \quad \frac{r'}{\sin F'NP} = \frac{F'N}{\sin F'PN}$$
$$= \frac{er}{\sin FPN} \qquad\qquad = \frac{er'}{\sin F'PN}$$

But $\sin FNP = \sin F'NP$

$\therefore \sin FPN = \sin F'PN$

CHAPTER VIII.

THE GENERAL EQUATION OF THE SECOND DEGREE.

187. The most general equation of the second degree between two variables x and y may be written in the form

$$mx^2 + ny^2 + 2lxy + 2px + 2qy + d = 0; \qquad (1)$$

the six coefficients m, n, l, p, q and d being any constants whatever.*

We may divide the equation throughout by any one of the coefficients without changing the relation between x and y. One of the six coefficients will then be reduced to unity. Hence the six coefficients are really equivalent to but five independent quantities.

The problem now before us is: What possible curves may be the locus of the general equation, and what common properties have these curves?

One property may be recognized at once by determining the points of intersection of the curve with a straight line. Let the equation of the straight line be

$$y = hx + b.$$

By substituting this value of y in (1) we shall have an equation of the second degree in x whose roots will give the abscissas of the points of intersection. Now, since an equation of the second degree always has two roots which may be real, equal or imaginary, we conclude:

---

* Three of these terms are written with the coefficient 2 because many expressions which enter into the theory, especially when determinants are introduced, are thus simplified. We then consider l, p and q as representing *one half* the coefficients of xy, x and y respectively.

THEOREM I. *Every straight line intersects a curve of the second degree in two real, coincident, or imaginary points.*

188. *Change of Origin.* To continue the investigation, we change the origin of co-ordinates without changing the form of the curve. If we put x' and y' for the co-ordinates referred to the new origin, we must, in equation (1), put

$$x = x' + a',$$
$$y = y' + b';$$

a' and b' being the co-ordinates of the new origin, which are to be determined in such a way as to simplify the equation. Making this substitution, the equation becomes

$$\begin{aligned} mx'^2 + 2lx'y' + ny'^2 + 2(a'm + b'l + p)x' + 2(a'l + b'n + q)y' \\ + a'^2m + b'^2n + 2a'b'l + 2a'p + 2b'q + d = 0. \qquad (2) \end{aligned}$$

We now so determine the co-ordinates a' and b' that the coefficients of x' and y' shall vanish. To effect this we have the equations

$$\left.\begin{aligned} ma' + lb' &= -p; \\ la' + nb' &= -q; \end{aligned}\right\} \qquad (a)$$

in which a' and b' are the unknown quantities. Solving the equations, we find

$$\left.\begin{aligned} a' &= \frac{np - lq}{l^2 - mn}; \\ b' &= \frac{mq - lp}{l^2 - mn}. \end{aligned}\right\} \qquad (3)$$

Omitting for the present the special case in which $l^2 - mn = 0$, these values of a' and b' will always be finite.

By means of these values of a' and b' we may simplify the equation (2) as follows: Multiplying the first of equations (a) by a', the second by b', and adding, we have

$$ma'^2 + nb'^2 + 2la'b' = -a'p - b'q. \qquad (b)$$

By means of the equations (a) and (b) the general equation (2), omitting accents, is reduced to

$$mx^2 + 2lxy + ny^2 + a'p + b'q + d = 0. \qquad (4)$$

This equation (4) will now represent the same curve as (1), only referred to new axes of co-ordinates.

189. A second fundamental property of the locus of the second degree is immediately deducible from (4). If x and y be any values of the co-ordinates which satisfy this equation, it is evident that $-x$ and $-y$ will also satisfy it. That is, if the point (x, y) lie on the curve, the point $(-x, -y)$ will also lie upon it. But the line joining these two points passes through the origin and is bisected by the origin. When referred to the original system (1), this origin is the point whose co-ordinates are a' and b' in (3). Hence:

THEOREM II. *For every curve of the second degree there is a certain point which bisects every chord of the curve passing through it.*

Def. The point which bisects every chord passing through it is called the **centre** of the curve.

REMARK 1. In the special case when

$$l^2 - mn = 0,$$

the centre (a', b') of the curve will be at infinity, and the theorem will not be directly applicable.

REMARK 2. Since in the equation (4) the origin is at the centre, this equation is that of the general curve of the second degree referred to its centre as the origin.

190. *Change of Direction of the Axes of Co-ordinates.* The next simplification of the equation will consist in removing the term in xy. To do this, let us refer the curve to the same origin as in (4), namely, the centre, but to a new system of axes making an angle δ with those of the original system. This we do by the substitution (§ 27)

$$x = x' \cos \delta - y' \sin \delta;$$
$$y = x' \sin \delta + y' \cos \delta.$$

Making this substitution, the equation becomes

$$\begin{aligned}&(m \cos^2\delta + n \sin^2\delta + 2l \sin \delta \cos \delta)x'^2 \\ +&(m \sin^2\delta + n \cos^2\delta - 2l \sin \delta \cos \delta)y'^2 \\ +&[(n - m) \sin 2\delta + 2l \cos 2\delta]x'y' = d',\end{aligned}$$

where we put $\qquad -d' \equiv a'p + b'q + d. \qquad$ (5)

Substituting for the powers and products of sines and cosines their values, namely,

$$\cos^2\delta = \tfrac{1}{2}(1 + \cos 2\delta),$$
$$\sin^2\delta = \tfrac{1}{2}(1 - \cos 2\delta),$$
$$2\sin\delta\cos\delta = \sin 2\delta,$$

and then putting, for brevity,

$$\left.\begin{aligned} h &\equiv (n-m)\cos 2\delta - 2l\sin 2\delta, \\ k &\equiv (n-m)\sin 2\delta + 2l\cos 2\delta, \end{aligned}\right\} \qquad (c)$$

this equation reduces to

$$(m+n-h)x'^2 + (m+n+h)y'^2 + 2kx'y' = 2d'. \qquad (6)$$

To make the term in $x'y'$ disappear, we must so determine the value of δ that $k = 0$. This gives

$$\tan 2\delta = \frac{2l}{m-n},$$

which determines the values of δ.

Then from (c) we have, when $k = 0$,

$$h\sin 2\delta - k\cos 2\delta = -2l \quad = h\sin 2\delta;$$
$$h\cos 2\delta + k\sin 2\delta = n - m = h\cos 2\delta.$$

The values of h and δ are therefore given by the equations

$$\left.\begin{aligned} h\sin 2\delta &= -2l; \\ h\cos 2\delta &= n-m; \end{aligned}\right\} \qquad (7)$$

whence

$$h = \sqrt{(m-n)^2 + 4l^2}. \qquad (8)$$

Omitting accents, the equation (6) of the curve now reduces to

$$\frac{m+n-\sqrt{(m-n)^2+4l^2}}{2d'}x^2 + \frac{m+n+\sqrt{(m-n)^2+4l^2}}{2d'}y^2 = 1. \qquad (9)$$

The coefficients of x^2 and y^2 in this equation are always real, but may be either positive or negative according to the sign of d' and the values of m, n and l.

If, then, we put

$$\left.\begin{aligned} \pm a^2 &\equiv \frac{2d'}{m + n - \sqrt{(m-n)^2 + 4l^2}}, \\ \pm b^2 &\equiv \frac{2d'}{m + n + \sqrt{(m-n)^2 + 4l^2}}, \end{aligned}\right\} \qquad (10)$$

the algebraic signs being so taken that a^2 and b^2 without sign shall be positive, the equation (9) still further reduces to

$$\pm \frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1, \qquad (11)$$

which still represents the same locus as (1), only referred to different axes and a different origin.

191. There are now three cases to be considered·

CASE I. The algebraic signs in the first member both negative.

CASE II. The algebraic signs both positive.

CASE III. The one sign positive and the other negative.

In the first case the equation is impossible with any real values of x and y, because the first member will then be necessarily negative, while the second is positive. The curve is therefore wholly imaginary.

In the second case the equation is that of an ellipse whose semi-axes are a and b.

In the third case the equation is that of an hyperbola whose semi-axes are a and b.

We therefore conclude:

THEOREM III. *The locus of the equation of the second degree between rectangular co-ordinates is a conic section.*

192. *Special Kinds of Conic Sections.* In order that the equation (9) shall represent an ellipse we must have, by Case II.,

$$m + n > \sqrt{(m-n)^2 + 4l^2}.$$

Hence $\qquad (m+n)^2 > (m-n)^2 + 4l^2$

and $\qquad mn > l^2,$

or $\qquad mn - l^2$ positive.

Hence:

THEOREM IV. *The criterion whether the general equation* (1) *shall represent an ellipse or an hyperbola is given by comparing the square of the coefficient of xy with four times the product of the coefficients of x^2 and y^2.*

If the square is algebraically less than four times the product, the curve is an ellipse; if greater, it is an hyperbola.

In special cases the equation may represent other lines than the ellipse or hyperbola. We have, in fact, tacitly assumed that the expressions a^2 and b^2 in (10) are both finite and determinate. We have now to consider the case when either of them is zero, infinity or indeterminate.

193. *The Parabola.* If in the equation (1) $mn = l^2$, the preceding criterion will give neither a genuine ellipse nor hyperbola, but a limiting curve between the two. We know the parabola to be such a curve. In this case, also, the co-ordinates a' and b' of the centre of the curve in (3) will be infinite, so that the equation cannot be reduced to the form (4). But when the centre of an ellipse or hyperbola recedes to infinity, we know from Elementary Geometry that the curve becomes a parabola. We shall now prove this result analytically.

Reduction in the Case of a Parabola. We have to consider the special form of the general equation (1) in which $l = \sqrt{mn}$. The equation may then be written in the form

$$(m^{\frac{1}{2}}x + n^{\frac{1}{2}}y)^2 + 2px + 2qy + d = 0. \qquad (12)$$

That is, in this case the sum of the three terms of the second order forms the square of the linear expression $m^{\frac{1}{2}}x + n^{\frac{1}{2}}y$. We may infer that the line whose equation is

$$m^{\frac{1}{2}}x + n^{\frac{1}{2}}y = 0$$

stands in some special relation to the curve. We shall therefore so change the direction of the axes of co-ordinates that this line shall be the new axis of X. Taking the general equation for this transformation,

$$\left.\begin{aligned} x &= x'\cos\delta - y'\sin\delta, \\ y &= x'\sin\delta + y'\cos\delta, \end{aligned}\right\} \qquad (a)$$

we see that they give

$$m^{\frac{1}{2}}x + n^{\frac{1}{2}}y = (m^{\frac{1}{2}}\cos\delta + n^{\frac{1}{2}}\sin\delta)x' + (-m^{\frac{1}{2}}\sin\delta + n^{\frac{1}{2}}\cos\delta)y'. \quad (b)$$

In order, now, that the line $m^{\frac{1}{2}}x + n^{\frac{1}{2}}y = 0$ may be identical with the line $y' = 0$ (which is the axis of X'), the coefficient of x' in the above equation must vanish. That is, we must have

$$m^{\frac{1}{2}}\cos\delta + n^{\frac{1}{2}}\sin\delta = 0, \quad (c)$$

or

$$\tan\delta = -\frac{m^{\frac{1}{2}}}{n^{\frac{1}{2}}};$$

whence

$$\sin\delta = \frac{\tan\delta}{\sqrt{1+\tan^2\delta}} = -\frac{m^{\frac{1}{2}}}{\sqrt{m+n}};$$

$$\cos\delta = \frac{1}{\sqrt{1+\tan^2\delta}} = +\frac{n^{\frac{1}{2}}}{\sqrt{m+n}};$$

$$-m^{\frac{1}{2}}\sin\delta + n^{\frac{1}{2}}\cos\delta = (m+n)^{\frac{1}{2}}. \quad (d)$$

(b) and (a) now become

$$m^{\frac{1}{2}}x + n^{\frac{1}{2}}y = \sqrt{(m+n)}\,y';$$

$$x = \frac{n^{\frac{1}{2}}x' + m^{\frac{1}{2}}y'}{\sqrt{(m+n)}}; \qquad y = \frac{n^{\frac{1}{2}}y' - m^{\frac{1}{2}}x'}{\sqrt{(m+n)}}.$$

By substitution, the equation (12) now becomes

$$(m+n)y'^2 + \frac{2pn^{\frac{1}{2}} - 2qm^{\frac{1}{2}}}{\sqrt{(m+n)}}x' + \frac{2pm^{\frac{1}{2}} + 2qn^{\frac{1}{2}}}{\sqrt{(m+n)}}y' + d = 0;$$

and putting, for brevity,

$$\tfrac{1}{2}P \equiv \frac{qm^{\frac{1}{2}} - pn^{\frac{1}{2}}}{(m+n)^{\frac{3}{2}}}, \quad (13)$$

$$\tfrac{1}{2}Q \equiv \frac{pm^{\frac{1}{2}} + qn^{\frac{1}{2}}}{(m+n)^{\frac{3}{2}}},$$

$$D \equiv \frac{d}{m+n},$$

the equation reduces to

$$y'^2 + Qy' - Px' + D = 0.$$

This, again, can be expressed in the form

$$(y' + \tfrac{1}{2}Q)^2 = \tfrac{1}{4}Q^2 + Px' - D. \tag{15}$$

We can still further simplify this equation by changing the origin to the point whose co-ordinates are $-\frac{1}{2}Q$ and $-\dfrac{\frac{1}{4}Q^2 - D}{P}$. If the new co-ordinates referred to this origin are x and y, we have

$$x = x' + \frac{\frac{1}{4}Q^2 - D}{P};$$

$$y = y' + \tfrac{1}{2}Q.$$

Then, by substitution, the equation becomes

$$y^2 = Px, \tag{16}$$

which is the equation of a parabola whose parameter is $\cancel{\tfrac{1}{2}}P$ referred to its vertex and principal axis.

We therefore conclude:

THEOREM V. *The general equation of the second degree represents a parabola when the square of the coefficient of xy is equal to four times the product of the coefficient of x^2 and y^2.*

194. *Case when the Parameter is Zero.* There is still a special case of the parabola to be considered, namely, that in which $P = 0$. From (16) it would then follow that $y = 0$ for all values of x. But this conclusion would be premature, because the transformation (15) would then involve the placing of a new origin at infinity. We must therefore go back to the equation (14), which, when $P = 0$, gives

$$y = -\tfrac{1}{2}Q \pm \sqrt{\tfrac{1}{4}Q^2 - D};$$

that is, y may have either of two constant values.

Hence, when $P = 0$, the equation represents a pair of straight lines parallel to the axis of X and distant $\sqrt{\frac{1}{4}Q^2 - D}$ on each side of it.

195. *General Case of a Pair of Straight Lines.*

On reducing to the form (4), the absolute term d' may vanish. The reduction to the form (9) will then be impossi-

ble, because the coefficients of x^2 and y^2 will become infinite. In this case, however, the equation (4) will be

$$mx^2 + 2lxy + ny^2 = 0. \tag{17}$$

If we factor this quadratic equation by any of the methods explained in Algebra, we may reduce it to the form

$$\{ny + (l + \sqrt{l^2 - mn})x\} \times \{ny + (l - \sqrt{l^2 - mn})x\} = 0,$$

or we may prove this equation by executing the indicated multiplications and thus reducing it to the form (17).

Now this equation may be satisfied by equating either of its two factors to zero. If we distinguish the values of y in the two factors by subscript indices, we may have either

$$\left.\begin{aligned} y_1 &= \frac{-l + \sqrt{l^2 - mn}}{2n}x \\ \text{or} \qquad y_2 &= \frac{-l - \sqrt{l^2 - mn}}{2n}x; \end{aligned}\right\} \tag{18}$$

that is, to each value of x will correspond these two values of y. But each equation (18) is that of a straight line passing through the origin. We therefore conclude:

THEOREM VI. *When, on reducing the general equation of the second degree to the centre, the absolute term vanishes, the equation represents a pair of straight lines.*

If we have $l^2 < mn$, the lines will both be imaginary. But in this case there will be one pair of real values of the coordinates, namely, $x = 0$ and $y = 0$. Hence,

If, in the case supposed in the preceding theorem, the lines become imaginary, the equation can be satisfied by only a single real point.

This result is also evident by a comparison of equations (9), (10) and (11), because when $d' = 0$ and $l^2 < mn$, we have an ellipse of which both the axes are zero, and this can be nothing but a point.

On the other hand, if both the axes of an hyperbola become zero, it reduces to a pair of straight lines.

We have thus found two seemingly distinct cases in

which the conic is reduced to two straight lines: the one when $l^2 - mn = 0$ and $P = 0$; the other when $d' = 0$. We shall now show that the former cases may be combined with the latter.

If in the expression (5) for d' we substitute for a' and b' their values (3), it will become

$$-d' = \frac{p(np - lq) + q(mq - lp) + d(l^2 - mn)}{l^2 - mn}.$$

Now let us put

$$R \equiv p(np - lq) + q(mq - lp) + d(l^2 - mn), \qquad (19)$$

so that we have

$$R = d'(mn - l^2). \qquad (20)$$

If we square the value (13) of P and note that we are considering the case when $mn = l^2$, we have

$$\begin{aligned}(m + n)^3P^2 &= mq^2 - 2lpq + np^2 \\ &= p(np - lq) + q(mq - lp).\end{aligned}$$

This expression is zero, by hypothesis, since $P = 0$. Comparing it with (19) and noting that $l^2 - mn = 0$, we see that the value of R vanishes in this case as it does when $d' = 0$. We therefore conclude that $R = 0$ is the condition that the conic shall reduce to a pair of straight lines.

196. *Summary of Conclusions.* The various conclusions which we have reached may be recapitulated as follows:

The general equation of the second degree,

$$mx^2 + 2lxy + ny^2 + 2px + 2qy + d = 0,$$

represents

An ellipse when $l^2 < mn$;
A parabola when $l^2 = mn$;
An hyperbola when $l^2 > mn$.

Also, in special cases,

The ellipse may be reduced to a point;
The parabola to a pair of parallel straight lines;
The hyperbola to a pair of intersecting straight lines.

But since, in the first case, the point-ellipse is defined as the real intersection of a pair of imaginary straight lines, we

may describe all three of these cases as one in which the conic is reduced to a pair of straight lines, and sum up the conclusion thus:

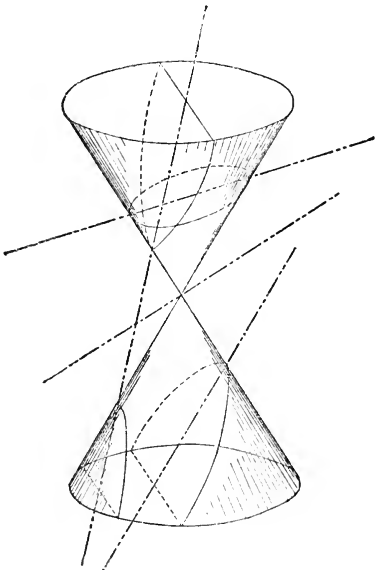
If the coefficients in the general equation of a conic satisfy the condition

$$p(np - lq) + q(mq - lp) + d(l^2 - mn) = 0, \qquad (19)$$

the conic will be reduced to a pair of straight lines. If we have

$l^2 < mn$, *the lines are imaginary;*
$l^2 = mn$, *the lines are real, and parallel or coincident;*
$l^2 > mn$, *the lines are real and intersecting.*

197. All these forms are conic sections. That the ellipse, parabola and hyperbola are such sections is shown in Geometry.



When the cutting plane passes through the vertex of the cone, the section is a point or a pair of intersecting straight lines according to the position of the plane.

When the vertex of the cone recedes to infinity, the base remaining finite, the cone becomes a cylinder, and the section parallel to the elements is a pair of parallel straight lines.

REMARK. A conic section is, for brevity, frequently called **a conic** simply, and we therefore designate all loci of the second degree as *conics.*

198. *Similar Conics.* From equation (10) it follows that the ratio $a : b$ of the semi-axes depends only upon the coefficients m, n and l of the terms of the second order in the general equation. Since we have, for the eccentricity,

$$e^2 = 1 \pm \frac{b^2}{a^2},$$

it follows that the eccentricity depends only on the same coefficients, l, m and n.

Moreover, the angle δ which the principal axes of the conic form with the original axes of co-ordinates depends only on these same coefficients. Hence, using the definitions,

Similar conics are those which have the same eccentricity or (which is the same thing) the same ratio of the two principal axes; Similar conics are said to be **similarly placed** when their corresponding axes are parallel,—we have the theorem:

THEOREM VII. *All conics whose equations have the same terms of the second degree in the co-ordinates are similar and similarly placed.*

199. THEOREM VIII. *A conic section may be made to pass through any five points in a plane.*

Let us divide the general equation (1) by d, and, distinguishing the new coefficients by accents, we have

$$m'x^2 + n'y^2 + 2l'xy + 2p'x + 2q'y + 1 = 0. \qquad (a)$$

Now if (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), (x_5, y_5) are the five given points in the plane, we have, by substituting in the last equation the co-ordinates of these five points for the general co-ordinates x and y, the five following equations of condition:

$$m'x_1^2 + n'y_1^2 + 2l'x_1y_1 + 2p'x_1 + 2q'y_1 + 1 = 0;$$
$$m'x_2^2 + n'y_2^2 + 2l'x_2y_2 + 2p'x_2 + 2q'y_2 + 1 = 0;$$
$$m'x_3^2 + n'y_3^2 + 2l'x_3y_3 + 2p'x_3 + 2q'y_3 + 1 = 0;$$
$$m'x_4^2 + n'y_4^2 + 2l'x_4y_4 + 2p'x_4 + 2q'y_4 + 1 = 0;$$
$$m'x_5^2 + n'y_5^2 + 2l'x_5y_5 + 2p'x_5 + 2q'y_5 + 1 = 0;$$

from which the coefficients m', n', l', p' and q' may be found, since x_1, y_1, x_2, y_2, etc., are *known quantities.* Substituting these values of m', n', etc., in the general equation (a), the resulting equation of the second degree in x and y will be that of the required conic section.

Cor. Since the equations of condition are all of the first degree with respect to m', n', l', p' and q', each of these quantities has only one value; therefore *only one conic section can be passed through five given points on a plane.*

EXAMPLE. Let it be required to pass a conic section through the five points $(2, 1)$, $(-1, -3)$, $(0, 3)$, $(1, 0)$, $(3, -2)$.

The equations of condition which determine the coefficients m, n, l, etc., are (omitting the accents)

$$\begin{array}{l} 4m + n + 4l + 4p + 2q + 1 = 0; \\ m + 9n + 6l - 2p - 6q + 1 = 0; \\ 9n + 6q + 1 = 0; \\ m + 2p + 1 = 0; \\ 9m + 4n - 12l + 6p - 4q + 1 = 0; \end{array}$$

from which we find

$$m = \frac{33}{64}; \quad n = -\frac{41}{192}; \quad l = -\frac{1}{32}; \quad p = -\frac{97}{128}; \quad q = \frac{59}{384}.$$

Substituting these values in the general equation

$$mx^2 + ny^2 + 2lxy + 2px + 2qy + 1 = 0,$$

and clearing of fractions, we have

$$99x^2 - 41y^2 - 12xy - 291x + 59y + 192 = 0,$$

which is the equation of an hyperbola, since $l^2 - mn$ is a positive quantity.

If one of the given points should be the origin, the corresponding equation would be the impossible one $1 = 0$. In this case we should have to divide by some other coefficient than $\bar{a}$.

200. *Intersection and Tangency of Conics.*

THEOREM IX. *Two conics in general intersect each other in four points.*

Proof. The co-ordinates x and y of the points of intersection of two conics are given by the roots of two equations of the second degree in x and y. Now, it is shown in Algebra that when we eliminate an unknown quantity from two quadratic equations, the resulting equation in the other unknown quantity will, in general, be of the fourth degree. This equation will therefore have four roots, thus giving rise to four sets of co-ordinates of the points of intersection.

REMARK. The roots may be all four real; one pair real and one pair imaginary; or all four imaginary; and, in any case, the two roots of a pair may be equal.

According as this happens the conics are said to intersect in *real, imaginary* or *coincident* points. In the latter case they are said to *touch* each other at the coincident points.

Cor. Two conics may touch each other at two points and no more.

201. *Families of Conics.* Let us put, for brevity,

$$\begin{aligned} P' &\equiv m'x^2 + 2l'xy + n'y^2 + 2p'x + 2q'y + d'\,; \\ P'' &\equiv m''x^2 + 2l''xy + n''y^2 + 2p''x + 2q''y + d''; \\ &\qquad \text{etc.} \qquad\qquad \text{etc.} \qquad\qquad \text{etc.}; \end{aligned}$$

that is, let us represent by P', P'', etc., any functions of the second degree in the co-ordinates.

THEOREM X. *If* $P' = 0$ *and* $P'' = 0$ *are the equations of any two different conics, then the equation*

$$\mu P' + \lambda P'' = 0 \qquad (20)$$

(*where* μ *and* λ *are constants*) *will represent a third conic passing through the four points of intersection of the other two.*

For, first, we see by substitution of the values of P' and P'' that the equation (20) is of the second degree in the co-ordinates. Hence its locus is *some* conic.

Secondly, every pair of values of x and y which make

both $P' = 0$ and $P'' = 0$ must also satisfy the equation $\mu P' + \lambda P'' = 0$. Hence every point common to P' and P'' must belong to the locus of (20); that is, this locus passes through all the points, real and imaginary, in which P' and P'' intersect. The number of these points is four.

By giving different values to the ratio $\lambda : \mu$, any number of conics passing through the same four points may be found. We may, without loss of generality, suppose $\mu = 1$ in this theory, because the locus (20) depends only on the ratio $\lambda : \mu$

Def. A system of conics all of which pass through the same four points is called a **family of conics.**

202. THEOREM XI. *In a family of conics two and no more are parabolas.*

Proof. If, in the expression

$$P = P' + \lambda P'',$$

we substitute for P' and P'' their values, we shall have, in P,

$$\begin{aligned} &\text{Coefficient of } x^2 = m' + \lambda m'' \equiv m; \\ &\text{Coefficient of } y^2 = n' + \lambda n'' \equiv n; \\ &\text{Coefficient of } 2xy = l' + \lambda l'' \equiv l. \end{aligned}$$

The condition that the curve $P = 0$ shall be a parabola then becomes

$$\begin{aligned} 0 &= l^2 - mn \\ &= (l''^2 - m''n'')\lambda^2 + (2l'l'' - m'n'' - m''n')\lambda + l'^2 - m'n'. \end{aligned}$$

This is a quadratic equation in λ, which therefore gives two values of λ, and thus two expressions for P, each of which, equated to zero, is the equation of a parabola. Q. E. D.

203. THEOREM XII. *In a family of conics three, and no more, may be pairs of lines.*

Proof. Forming the expression $P' + \lambda P''$, we find the coefficients of the general equation to become

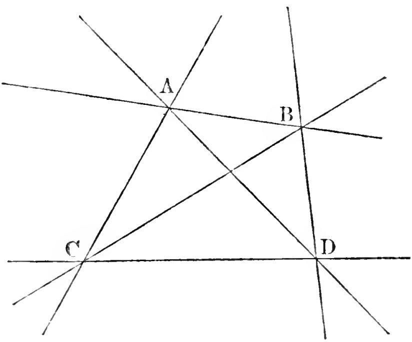
$$\begin{aligned} m &= m' + \lambda m''; \\ n &= n' + \lambda n''; \\ &\text{etc.} \qquad \text{etc.} \end{aligned}$$

In order that a conic of the family may be a pair of lines it is necessary and sufficient that its coefficients satisfy the condition (19). Each term of (19) is of the third degree in the coefficients. Hence the entire condition gives an equation of the third degree in λ, which has three roots. Hence we have three expressions of the form $P' + \lambda P''$, each of which, when equated to zero, gives a pair of lines. Q. E. D.

REMARK. If we call A, B, C and D the four points of intersection of the family, the three pairs of lines which belong to it will pass as follows:

One pair through AB and CD respectively;
One pair through AC and BD respectively;
One pair through AD and BC respectively;

and the three pairs will form the sides and diagonals of a quadrilateral.



204. THEOREM XIII. *If we take any point* (x_1, y_1) *at pleasure in the plane of a family of conics, then one conic of the family, and no more, will pass through this point.*

Proof. Since the equation

$$P' + \lambda P'' = 0$$

must be satisfied for the value (x_1, y_1) of the co-ordinates x and y which enter into it, we have

$$(m' + \lambda m'')x_1^2 + (n' + \lambda n'')y_1^2 + 2(l' + \lambda l'')x_1 y_1 + \text{etc.} = 0.$$

Since x_1, y_1 and all the symbols except λ in this equation are known, it is an equation of the first degree in λ and so has but one root. This root may be expressed in the form

$$\lambda = -\frac{P_1'}{P_1''},$$

in which P_1' and P_1'' represent the values of P' and P'' when x_1 and y_1 are substituted for x and y. There being but one value of λ, only one conic of the family can pass through the point (x_1, y_1).

205. THEOREM XIV. *The equation*

$$P' + \lambda P'' = 0 \tag{a}$$

may, by giving all real values to λ, *represent every possible conic passing through the four intersections of* P' *and* P''.

For, let C be any conic passing through the four points. Take any fifth point (x_1, y_1) on C, and put $\lambda = -\frac{P_1'}{P_1''}$. The equation (a) will then be satisfied identically when in it we put

$$x = x_1, \qquad y = y_1,$$

because it will become

$$P_1' - \frac{P_1'}{P_1''}P_1'' = 0.$$

Hence, with this value of λ, (a) will represent a conic of the family passing through the point (x_1, y_1). But only one conic can pass through five points. Hence the conic thus found will be C.

206. *Relation of Focus and Directrix to the General Equation.* Let BAC be any conic section; OX, OY, rectangular axes. Let AQ, the axis of the curve, make an angle $AGX \equiv \alpha$ with the axis OX. And let (x, y) be the co-ordinates of any point P; (h, k), the co-ordinates of the focus F; and $r = OD$, the distance from the origin to the directrix DK. Join PF, and draw PE perpendicular to DK, and

FH parallel to OX. Then, by the definition of a conic section given in § 181, Chapter VII., we have

$$\frac{PF}{PE} = e, \text{ the eccentricity,}$$

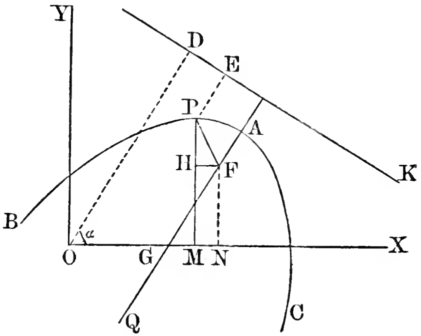
and therefore $$PF = ePE$$

and $$PE = x \cos \alpha + y \sin \alpha - r.$$

Hence $$PF = e(x \cos \alpha + y \sin \alpha - r),$$

$$PF^2 = FH^2 + PH^2,$$

or $$(x \cos \alpha + y \sin \alpha - r)^2 e^2 = (x - h)^2 + (y - k)^2.$$



Expanding and collecting terms, we have

$$(1 - e^2 \cos^2\alpha)x^2 + (1 - e^2 \sin^2\alpha)y^2 - 2e^2 \sin \alpha \cos \alpha \, xy \\ + (2e^2 r \cos \alpha - 2h)x + (2e^2 r \sin \alpha - 2k)y \\ + (h^2 + k^2 - e^2 r^2) = 0. \qquad (21)$$

To compare this with the general equation, we must divide both it and the general equation by their absolute terms, in order that the two may have the same coefficients. Supposing the general equation thus divided, and writing m for $\frac{m}{d}$, n for $\frac{n}{d}$, etc.: also putting, for brevity,

$$\lambda \equiv \cos \alpha; \qquad \mu \equiv \sin \alpha;$$

we find, by comparing coefficients of corresponding terms in the two equations,

$$\frac{1 - e^2\lambda^2}{h^2 + k^2 - e^2r^2} = m,$$

or

$$1 - e^2\lambda^2 = m(k^2 + h^2 - e^2r^2); \qquad (a)$$

$$1 - e^2\mu^2 = n(k^2 + h^2 - e^2r^2); \qquad (b)$$

$$- e^2\lambda\mu = l(k^2 + h^2 - e^2r^2); \qquad (c)$$

$$e^2\lambda r - h = p(k^2 + h^2 - e^2r^2); \qquad (d)$$

$$e^2\mu r - k = q(k^2 + h^2 - e^2r^2). \qquad (e)$$

These five equations completely determine the five quantities α, r, h, k and e, and hence the focus, directrix and eccentricity of the conic, in terms of the coefficients of the general equation.

EXERCISES.

1. Investigate the locus represented by the equation

$$4x^2 + y^2 + 3xy - 2x + y = 0.$$

Here we have $\quad m = 4; \quad n = 1; \quad l = \frac{3}{2}.$

Then $$mn - l^2 = 4 - \frac{9}{4} = + \frac{7}{4};$$

therefore the locus is an ellipse.

2. Find the co-ordinates of the centre of the conic represented by

$$5x^2 + y^2 + 2xy - 37x - 2y + 100 = 0,$$

and find the angle between the axis of the curve and the axis of X.

3. What curve does $y^2 = 3(xy - 2)$ represent?

4. Determine the locus $y^2 = 3(x - 7)$ and the angle its axis makes with the axis of X.

5. Determine the locus of $x^2 + y^2 - 6xy - 6x + 2y + 5 = 0$. Find co-ordinates of the centre, and the angle the axis of the curve makes with the axis of X. *Ans.* $(0, -1)$; $135°$.

6. If $Ay^2 + Bxy + Cx^2 + Dy + Ex + F = 0$ be the equation of a conic section, show that

$$Bx + 2Ay + D = 0$$

is the equation of a diameter of the locus.

7. From the equation (9) find the two conditions that the equation of the second degree shall represent a circle.

8. Find in the same way the two conditions that the general equation shall represent an equilateral hyperbola.

9. What locus is represented by the equation

$$h^2x^2 + mxy + k^2y^2 = c^2,$$

when $m = hk$?

10. Find the semi-parameter of the parabola

$$(x - y)^2 = ax.$$

11. What angle do the asymptotes of the hyperbola

$$mx^2 - xy = a$$

make with the transverse axis?

12. If we have the two conics

$$mx^2 + 2lxy + ny^2 + 2px + 2qy + d = 0,$$
$$mx^2 + 2lxy + ny^2 - 2px - 2qy + d = 0,$$

show that the line joining their centres is bisected by the origin.

13. The co-ordinates x and y of a moving point are expressed in terms of the time t by the equations

$$x = mt + a; \qquad y = mt + b.$$

What is the equation of the line described by the point?

14. If the co-ordinates are given by the equations

$$x = mt, \qquad y = nt^2,$$

show that the curve is a parabola, and express its parameter.

15. What condition must the coefficients of the general equation (1) of the second degree satisfy that the curve may pass through the origin of co-ordinates?

16. Write the equation of that conic formed of a pair of straight lines through the origin whose slopes are m and $-m$.

17. Do the same thing when the lines are to intersect in the point (a, b).

18. What is the condition that the principal axes of a conic shall be parallel to the axes of co-ordinates? (See §190.)

19. Express the points in which the locus of the equation

$$x^2 - 2xy + y^2 + 3x - 4 = 0$$

cuts the respective axes of co-ordinates.

Ans. The axis of X, $(1, 0)$ and $(-4, 0)$;
The axis of Y, $(0, 2)$, $(0, -2)$.

20. What condition must the coefficients of (1) satisfy that the curve may be tangent to the axis of X and to the axis of Y respectively?

Ans. $p^2 = md$ for the axis of X;
$q^2 = nd$ for the axis of Y.

The solution is very simple, if it is remembered that the curve is to cut the axis in two coincident points.

21. Find the equation of that conic which cuts the axis of X at points whose abscissas are -2 and $+4$, the axis of Y at points whose ordinates are -1 and $+2$, and whose principal axes are parallel to the axes of co-ordinates.

Ans. $x^2 + 4y^2 - 2x - 4y - 8 = 0$.

22. Show that in the general equation (1) the line

$$mx + ly - p = 0$$

bisects all chords parallel to the axis of X. Find also the line which bisects all chords parallel to the axis of Y.

Begin by solving the general equation as a quadratic in x so as to express x in terms of y, and *vice versa*.

23. How many points are necessary to determine a parabola? An equilateral hyperbola?

24. Mark five points at pleasure on a piece of paper. Can you find any criterion for distinguishing at sight the following cases?—

I. The five points lie on one branch of a conic (ellipse or hyperbola).

II. The conic is an hyperbola having three of the points on one branch and two on the other.

III. It is an hyperbola having four points on one branch and one on the other.

Suppose a string drawn tightly around all the points, and note the number of points the string will not reach.

25. Find the equation of a parabola which shall touch the axis of X at the point whose abscissa is $+2$, and the axis of Y at the point whose ordinate is $+1$.

Ans. $x^2 - 4xy + 4y^2 - 4x - 8y + 4 = 0$.

26. The base of a triangle has a fixed length, and the escribed circle below this base is required to touch it at a fixed point. Find the locus of the point of intersection of the two sides of the triangle.

27. A line passes through the fixed point $(0, b)$ on the axis of Y and intersects the axis of X and the fixed line $y = mx$. Find the locus of the middle point of the segment of the line contained between the fixed line and the axis of X.

28. Investigate the locus of the point the differences of the squares of whose distances from the axis of X and from the line $y = mx$ is the constant quantity k^2.

29. The base a of a triangle and the sum of the angles at the two ends of the base are both constant. Investigate the locus of its vertex.

30. Each abscissa of the circle $x^2 + y^2 = r^2$ is increased by m times its ordinate. Find the locus of the ends of the lines thus formed.

31. Investigate the locus of the middle points of all chords of an ellipse which pass through a fixed point.

32. The circle $x^2 + y^2 = r^2$ has two tangents intersecting in a movable point P and cutting out a fixed length a from a third tangent $y = r$. Investigate the locus of P.

33. Show that the equation of that pair of straight lines formed of the axes of co-ordinates is $xy = 0$.

PART II.

GEOMETRY OF THREE DIMENSIONS.

CHAPTER I.

POSITION AND DIRECTION IN SPACE.

207. *Directions and Angles in Space.* Two straight lines cannot form an angle, as that term is defined in elementary geometry, unless they intersect. Two lines in space will, in general, pass each other without intersecting. Hence we cannot speak of the angle between such lines unless we extend the meaning of the word *angle.* Now the following theorem is known from solid geometry:

If we have given any two lines, a and b, in space;

and if we take any point P at pleasure;

and if through P we draw two lines, PA and PB, parallel to a and b respectively,—

then, so long as we leave a and b unchanged, the angle APB will have the same value no matter where we take the point P.

We therefore take this angle as the *measure* of the angle between the lines a and b. This measure may be considered as expressing the *difference of direction* between the lines a and b, and the word *angle,* as applied to two non-intersecting lines, will be understood to mean their difference of direction. We thus have the following definition and corollary:

Def. The angle between two non-intersecting lines is measured by the angle between any two intersecting lines parallel to them.

Cor. *If we have two systems of parallel lines in space, the one a, a', a'', etc., the other b, b', b'', etc., then the angles between any line of a and any line of b will all be equal to each other.*

208. *Projections of Lines.* The **projection** of a finite line PQ upon an indefinite line X is the length of X intercepted by the perpendiculars dropped upon it from the two ends of PQ.

THEOREM I. *The projection of a line is equal to the product of its length by the cosine of the angle which it forms with the line on which it is projected.*

To prove the theorem, pass through each of the termini, P and Q, a plane perpendicular to X.* The planes will be parallel to each other and will cut X at the termini of the projection of PQ, which we may call P' and Q'. Through P' draw $P'Q'' \parallel PQ$ and intersecting the plane through Q in Q''. We shall then have

$$P'Q'' = PQ;$$

(being parallels between parallel planes;)

$$\begin{aligned} P'Q' &= P'Q'' \cos Q''P'Q' \\ &= PQ \cos (\text{angle between } P'Q' \text{ and } PQ). \quad \text{Q.E.D.} \end{aligned}$$

REMARK. By assigning a positive and negative direction to the two lines, the algebraic sign of the projection will be determined. It will be positive or negative according as the angle between the positive directions of the two lines is less or greater than a right angle. The following theorem is a result of this convention, combined with the principles of Trigonometry:

THEOREM II. *If we have any broken line in space, made up of the consecutive straight lines AB, BC, CD, etc., . . . GH, which lines form the angles α, β, γ, etc., with the line of projection X;*

and if we project this line upon X by dropping perpendiculars AA'.BB', CC', etc., ... HH', meeting X at the points A', B', C', D', etc., ... H'—

then the length $A'H'$ will be the algebraic sum of the

---

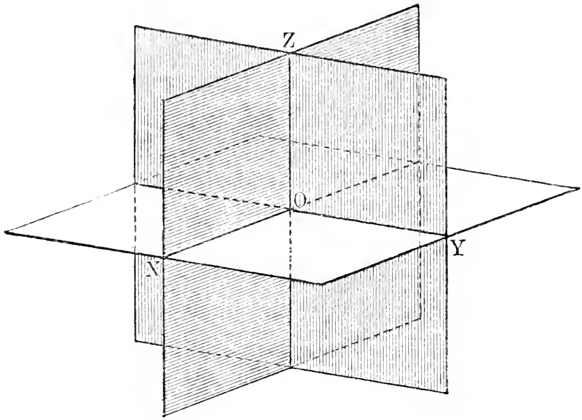
* No figure is drawn for this demonstration, because two non-intersecting lines in space cannot be represented on paper. If the student cannot readily conceive the relation, he should take two rods or pencils to represent the lines.

separate lengths $A'B'$, $B'C'$, $C'D'$, etc., these separate lengths being considered positive when taken in one direction, negative when taken in the opposite direction, and will be expressed by the equation

$$A'H' = AB \cos\alpha + BC \cos\beta + CD \cos\gamma + \text{etc.}$$

209. *Co-ordinate Axes and Planes in Space.* The position of a point in space may be defined by its relation to three straight lines intersecting in the same point and not lying in a plane.

Three such lines of reference are called a system of **co-ordinate axes** in space.



The point in which the axes intersect is called the **origin of co-ordinates,** or simply the **origin.**

The three axes are designated by the letters X, Y and Z respectively.

The co-ordinate axes, taken two and two, lie in three planes, one containing the two axes X and Y, another Y and Z, a third Z and X.

These planes are called **co-ordinate planes.** They are distinguished as the plane of XY, the plane of YZ, and the plane of ZX respectively.

The several angles which the axes of co-ordinates make with each other are arbitrary. But, for elementary purposes, it is most convenient to suppose each to form a right angle with

the other two. The following conclusions then result from solid geometry:

I. Each axis is perpendicular to the plane of the other two.

II. Each plane is perpendicular to the other two planes.

III. Every line or plane perpendicular to one of the planes is parallel to the axis which does not lie in that plane.

IV. Every line or plane perpendicular to one of the axes is parallel to the plane of the other two.

V. If the centre of a sphere lies in the origin, the intersections of the co-ordinate axes and planes with its surface form the vertices and sides of eight trirectangular spherical triangles.

VI. To each plane corresponds the axis perpendicular to it, which is therefore called the axis of the plane.

210. *Co-ordinates.* The position of a point in space is defined by its distances from the three co-ordinate planes of a system, each distance being measured on a line parallel to the axis of the plane. When the axes are rectangular, these directions will be perpendicular to the planes. The notation is:

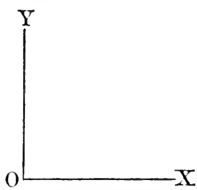
$$x \equiv \text{distance from plane } YZ;$$
$$y \equiv \text{distance from plane } ZX;$$
$$z \equiv \text{distance from plane } XY.$$

To distinguish between equal distances on the two sides of a plane, distances on one side are considered *positive*, on the other *negative*.

The positive direction from each plane is the positive direction of the axis perpendicular to it.

It is, of course, a matter of convention which side we take as positive and which negative. A certain relation between the positive directions is, however, adopted in physics and astronomy, and should be adhered to. It is this:

The positive side of the plane of XY is that from which we must look in order that the axis OX would have to turn in a direction the opposite of that of the hands of a watch in order to take the position OY.



If we conceive the plane of XY to be horizontal, the axis of Z will be vertical, and, supposing the

axes of X and Y to be arranged as in plane geometry, the positive side of the plane will be the upper one.*

The following propositions respecting certain relations of signs of co-ordinates to position should be perfectly clear to the student:

I. The co-ordinate planes divide the space surrounding the origin into eight regions, distinguished by the distribution of $+$ and $-$ signs among the co-ordinates.

Imagine the axis of X to go out positively toward the east;
Imagine the axis of Y to go out positively toward the north;
Imagine the axis of Z to go out positively upward,
and the point of reference to be the origin. Then—

II. For all points above the horizon z will be positive; for all points below it, negative.

III. For all points east of the north and south line x will be positive; for all points west of it, negative.

IV. For all points north of the east and west line y will be positive; for all points south of it, negative.

211. *How the Co-ordinates define Position.* Let us first suppose that the only information given us respecting the position of a point P is its co-ordinate

$$x = a, \qquad (a)$$

a being a given quantity.

This is the same as saying that P is at a distance a from the plane YZ. In order that a point may be at a distance a from a plane, it is necessary and sufficient that it lie in a parallel plane, such that the distance between the two planes is a.

Hence the proposition informs us that P lies in a certain plane.

---

* The author regards it as unfortunate that many mathematical writers, in treating of analytic geometry, reverse the arrangement of axes in space universally adopted in astronomy and physics. Uniformity in this respect is so desirable that he has not hesitated to adhere to the latter arrangement.

It may be remarked that, in drawing figures, the axes are represented as seen from different stand-points in different problems, the best point of view for each individual problem being chosen.

If we are informed that

$$y = b,$$

then P must lie in a plane parallel to ZX, at the distance b from it.

If both propositions, $x = a$ and $y = b$, are true, then P must lie in both planes. Hence it must lie on their line of intersection, which line will be

parallel to the axis of Z,
parallel to the planes ZX and YZ,
and perpendicular to plane XY.

If it is also added that

$$z = c,$$

the point P lies in a third plane, parallel to XY. Lying in all three planes, its only position will be their common point of intersection.

Hence:

The position of a point is completely determined when its three co-ordinates are given.

NOTATION. By point (a, b, c) we mean the point for which $x = a$, $y = b$ and $z = c$.

212. *Parallelopipedon formed by the Co-ordinates.*

Let O be the origin; OX, OY, OZ, the axes; P, the point; PR, PS and PQ, parallels to the axes terminating in the several planes. Then, by definition, the co-ordinates of P will be

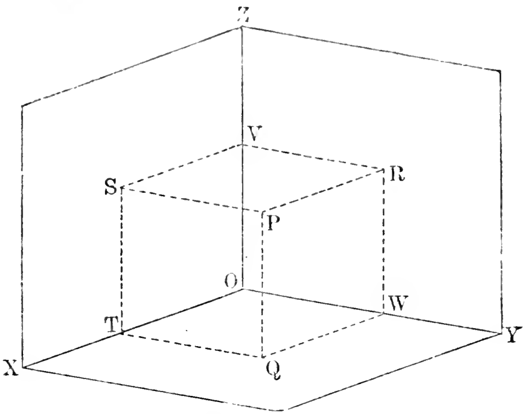
$$\begin{aligned} x &= RP = VS = OT = WQ; \\ y &= SP = VR = OW = TQ; \\ z &= QP = TS = OV = WR. \end{aligned}$$

We shall then have, by considering the three planes which contain these co-ordinates,

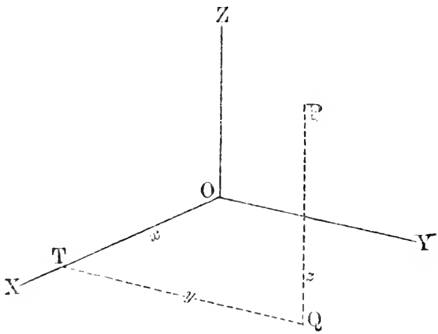
$$\begin{aligned} &\text{Plane } RPS \parallel \text{plane } XY; \\ &\text{Plane } SPQ \parallel \text{plane } YZ; \\ &\text{Plane } QPR \parallel \text{plane } ZX. \end{aligned}$$

Hence the three planes which contain the co-ordinates, together with those which contain the axes, form the six faces of a parallelopipedon. This figure has

Four edges = and || to co-ordinate x;
Four edges = and || to co-ordinate y;
Four edges = and || to co-ordinate z.



213. Since there are four equal lines for each co-ordinate, we may use any one of these four in constructing the co-ordinate. Sometimes it is advantageous to choose such lines that, taking the co-ordinates in some order, the end of each shall coincide



with the beginning of the next following, the end of the third being the point. For example, we may take, in order,

$$x = OT;$$
$$y = TQ;$$
$$z = QP.$$

The three co-ordinates will then form a series of three lines, each at a right angle with the other two.

Again, each face of the parallelopipedon being perpendicular to four edges, it follows that the diagonal PT will be the perpendicular from P upon the axis of X, and that the like proposition will be true for the other axes. Hence

The rectangular co-ordinates of a point are equal to the segments of the axes contained between the origin and the perpendiculars dropped from the point upon the respective axes.

EXERCISES.

1. If from the point (a, b, c) we draw lines to the several points $(-a, b, c)$, $(a, -b, c)$, $(a, b, -c)$, $(-a, -b, c)$, $(a, -b, -c)$, $(-a, b, -c)$, $(-a, -b, -c)$, define in what seven points these lines will cut such of the co-ordinate planes as they intersect.

2. If perpendiculars be dropped from a point (a, b, c) upon the three co-ordinate axes, show that the lengths of the perpendiculars will be $\sqrt{a^2 + b^2}$, $\sqrt{b^2 + c^2}$ and $\sqrt{c^2 + a^2}$.

3. If we take, on each axis, a point at the distance r from the origin, what will be the mutual distances of the three points from each other, and of what figure will they and the origin form the vertices?

$2 \cos \alpha = OL$
$r \sin \alpha = OL$
$\therefore$
$= \frac{1}{OR^2} + \frac{1}{OQ^2}$

4. If, on the axes of X, Y and Z respectively, we take the points P, Q and R, and from the origin O drop $OL \perp QR$, $OM \perp RP$ and $ON \perp PQ$, show that

$$\frac{1}{OL^2} + \frac{1}{OM^2} + \frac{1}{ON^2} = \frac{2}{OP^2} + \frac{2}{OQ^2} + \frac{2}{OR^2}.$$

214. PROBLEM. *To find the distance of a point (x, y, z) from the origin, and the angles which the line joining it to the origin makes with the co-ordinate axes.*

Let P be the point, and let us put

r, the distance OP from the origin;

α, β, γ, the angles POX, POY and POZ

which the line makes with the axes.

Then—

I. Because OP is the diagonal of a rectangular parallelo-

pipedon whose edges are PQ, PR and PS, we have, by Geometry,

$$OP^2 = PQ^2 + PR^2 + PS^2;$$

that is,

$$r^2 = x^2 + y^2 + z^2 \qquad (1)$$

and

$$r = \sqrt{x^2 + y^2 + z^2},$$

which gives the distance of the point from the origin.

II. Again, supposing the same construction as in § 212, we have

$$PTO = \text{a right angle.}$$

Hence

$$OT = OP \cos POX,$$

or, from the equality of the parallel edges,

In the same way

$$\left.\begin{aligned} x &= r \cos \alpha. \\ y &= r \cos \beta; \\ z &= r \cos \gamma. \end{aligned}\right\} \qquad (a)$$

The required values of the cosines of the angles are, therefore,

$$\left.\begin{aligned} \cos \alpha &= \frac{x}{r} = \frac{x}{\sqrt{x^2 + y^2 + z^2}}; \\ \cos \beta &= \frac{y}{r} = \frac{y}{\sqrt{x^2 + y^2 + z^2}}; \\ \cos \gamma &= \frac{z}{r} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}. \end{aligned}\right\} \qquad (2)$$

THEOREM III. *The sum of the squares of the cosines of the angles which a line through the origin makes with three rectangular axes is unity.*

Proof. Adding the squares of the last set of equations, we have

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{x^2 + y^2 + z^2}{x^2 + y^2 + z^2} = 1, \qquad (b)$$

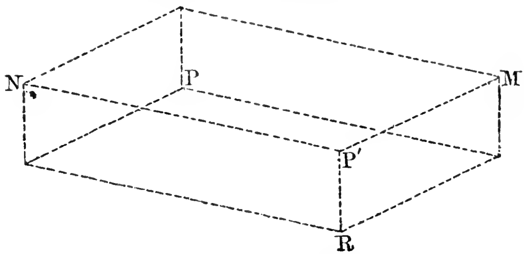
which proves the theorem.

This theorem enables us to find any one of the angles α, β and γ when the other two are given.

215. PROBLEM. *To express the distance between two points given by their co-ordinates, and the angles which the line joining them forms with the axes of co-ordinates.*

Let P and P' be the points, and let

P have the co-ordinates x, y, z;
P' have the co-ordinates x', y', z'.



Through each of the points P and P' pass three planes parallel to the co-ordinate planes. These planes will form the faces of a rectangular parallelopiped of which the edges are

$$P'M = x' - x;$$
$$P'N = y' - y;$$
$$P'R = z' - z.$$

If we put

$\varDelta \equiv PP'$, the distance of the points, we have, by Geometry,

$$\begin{aligned}\varDelta^2 &= P'M^2 + P'N^2 + P'R^2\\ &= (x' - x)^2 + (y' - y)^2 + (z' - z)^2\\ &= x'^2 + y'^2 + z'^2 + x^2 + y^2 + z^2 - 2(xx' + yy' + zz').\end{aligned}$$

Hence $\quad \varDelta = \sqrt{(x' - x)^2 + (y' - y)^2 + (z' - z)^2}. \qquad (3)$

To express the angles α, β and γ which the line PP' forms with the axes, we note that these angles are, by § 207, equal to those which $P'P$ forms with $P'M$, $P'N$ and $P'R$ respectively. Thus we find

$$\left.\begin{aligned}\cos\alpha &= \frac{x' - x}{\varDelta};\\ \cos\beta &= \frac{y' - y}{\varDelta};\\ \cos\gamma &= \frac{z' - z}{\varDelta}.\end{aligned}\right\} \qquad (4)$$

216. PROBLEM. *To express the angle between two lines in terms of the angles which each of them forms with the co-ordinate axes.*

Let the two lines emanate from the origin, and put

α, β, γ, the angles which one line makes with the axes;
α', β', γ', the corresponding angles for the other line.

On each line take a point, namely, P on the one and P' on the other, and put

r, r', the distances of P and P' respectively from the origin.

The problem is solved by expressing the distance PP' in two ways:

I. The equations (a) of § 214 give, for the co-ordinates

$$\begin{aligned}&\text{of } P: \ x = r \cos\alpha; \quad y = r \cos\beta; \quad z = r \cos\gamma;\\ &\text{of } P': \ x' = r' \cos\alpha'; \quad y' = r' \cos\beta'; \quad z' = r' \cos\gamma'.\end{aligned}$$

Substituting these values in the expression of § 215 for the distance of the points,

$$\begin{aligned}\Delta^2 &= r^2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma)\\ &\quad + r'^2(\cos^2\alpha' + \cos^2\beta' + \cos^2\gamma')\\ &\quad - 2rr'(\cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma').\end{aligned}$$

The first two terms reduce to $r^2 + r'^2$ by (b).

II. If we put

$$v \equiv \text{the angle between the lines,}$$

the lines r, r' and Δ will be the sides of a plane triangle of which the angle opposite the side Δ is v. Hence, by Trigonometry,

$$\Delta^2 = r^2 + r'^2 - 2rr' \cos v.$$

III. Comparing the two values of Δ^2, we find

$$\cos v = \cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma', \quad (5)$$

which is the required expression.

Cor. The condition that v shall be a right angle is

$$\cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma' = 0, \quad (6)$$

because, in order that an angle shall be a right angle, it is necessary and sufficient that its cosine shall be zero.

217. *Def.* The **direction-cosines** of a line are the cosines of the three angles which it forms with the co-ordinate axes.

The direction-cosines are so called because they determine the direction of the line.

Direction-Vectors. The three direction-cosines of a line are not independent, because, when any two are given, the third may be found by the equation

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1. \tag{7}$$

But *the direction of the line may be defined by any three quantities proportional to its direction-cosines.* To show this, let us put

$l, m, n,$ any three quantities proportional to $\cos\alpha$, $\cos\beta$, $\cos\gamma$ respectively.

Because of this proportionality, we shall have

$$\frac{l}{\cos\alpha} = \frac{m}{\cos\beta} = \frac{n}{\cos\gamma} \equiv \sigma.$$

whence

$$\sigma\cos\alpha = l; \qquad \sigma\cos\beta = m; \qquad \sigma\cos\gamma = n.$$

The sum of the squares of these equations gives, by (7),

$$\left.\begin{aligned} \sigma^2 &= l^2 + m^2 + n^2; \\ \cos\alpha &= \frac{l}{\sigma} = \frac{l}{\sqrt{l^2 + m^2 + n^2}}; \\ \cos\beta &= \frac{m}{\sigma} = \frac{m}{\sqrt{l^2 + m^2 + n^2}}; \\ \cos\gamma &= \frac{n}{\sigma} = \frac{n}{\sqrt{l^2 + m^2 + n^2}}. \end{aligned}\right\} \tag{8}$$

Thus, when l, m and n are given, the angles α, β and γ can be found, and thus the direction of the line is fixed.

The quantities l, m and n are called **direction-vectors.**

The direction of the line depends only upon the mutual ratios of the direction-vectors, and not upon their absolute values. For, if we multiply the three quantities l, m and n by any factor ρ, σ will be multiplied by this same factor, which will divide out from the equations (8), and thus leave the values of α, β and γ unchanged.

Cor. If the directions of two lines are given by the direction-vectors

$$l,\ m \text{ and } n, \qquad l',\ m' \text{ and } n',$$

respectively, the condition that they shall form a right angle is

$$ll' + mm' + nn' = 0. \tag{9}$$

For, by substituting in the equation (6) the values of the direction-cosines of the two lines given by (8), the condition becomes

$$\frac{ll' + mm' + nn'}{\sigma\sigma'} = 0,$$

from which (9) immediately follows.

218. PROBLEM. *To express the square of the sine of the angle between two lines in terms of their direction-cosines.*

The result is derived from (5) by the form

$$\sin^2 v = 1 - \cos^2 v = \cos^2\alpha + \cos^2\beta + \cos^2\gamma - \cos^2 v.$$

To simplify the writing we shall omit the letters *cos*, using α, β, γ and α', β', γ' for the direction-cosines of the respective lines. Then

$$\begin{aligned}
\sin^2 v &= \alpha^2 + \beta^2 + \gamma^2 - \cos^2 v \\
&= \alpha^2 + \beta^2 + \gamma^2 - \alpha^2\alpha'^2 - \beta^2\beta'^2 - \gamma^2\gamma'^2 \\
&\qquad\qquad - 2\alpha\alpha'\beta\beta' - 2\beta\beta' v\gamma' - 2\gamma\gamma'\alpha\alpha' \\
&= \alpha^2(1 - \alpha'^2) + \beta^2(1 - \beta'^2) + \gamma^2(1 - \gamma'^2) \\
&\qquad\qquad - 2(\alpha\alpha'\beta\beta' + \text{etc.}) \\
&= \alpha^2(\beta'^2 + \gamma'^2) + \beta^2(\gamma'^2 + \alpha'^2) + \gamma^2(\alpha'^2 + \beta'^2) \\
&\qquad\qquad - 2(\alpha\alpha'\beta\beta' + \text{etc.}) \\
&= \alpha^2\beta'^2 + \alpha'^2\beta^2 - 2\alpha\alpha'\beta\beta' + \beta^2\gamma'^2 + \beta'^2\gamma^2 - 2\beta\beta'\gamma\gamma' \\
&\qquad\qquad + \gamma^2\alpha'^2 + \gamma'^2\alpha^2 - 2\gamma\gamma'\alpha\alpha' \\
&= (\alpha\beta' - \alpha'\beta)^2 + (\beta\gamma' - \beta'\gamma)^2 + (\gamma\alpha' - \gamma'\alpha)^2,
\end{aligned} \tag{10}$$

which is the required expression.

Cor. If two lines have the direction-cosines of the one respectively equal to the corresponding ones of the other, the lines are parallel.

For, if

$$\alpha = \alpha', \qquad \beta = \beta', \qquad \gamma = \gamma',$$

the last equation reduces to

$$\sin v = 0.$$

EXERCISES.

1. If a line make equal angles with the co-ordinate axes, what are these angles, and what angle does it form with the co-ordinate planes? *Ans.* 54° 44′.1; 35° 15′.9.

2. Find the direction-cosines of a line which makes equal angles with the axes of X and Y, but double the common value of those equal angles with the axis of Z, and show that the angles may be either 90°, 90°, 180°; 45°, 45°, 90°; or 135°, 135°, 270°.

3. In a room 15 by 20 feet and 10 feet high, a line is stretched from the northwest corner of the ceiling to the southeast corner of the floor. Find its length, the angles which it forms with the three bounding edges of the walls and ceiling, and with the walls and ceiling.

4. What angles do lines having the following direction-vectors form with the co-ordinate axes?—

Line A, $l = 1$; $m = 2$; $n = 2$;
Line B, $l = 3$; $m = 2$; $n = 1$;
Line C, $l = 2$; $m = 3$; $n = 4$;
Line D, $l = p$; $m = 2p$; $n = 3p$.

5. If the direction-cosines of a line are proportional to the fractions $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{6}$, what are the smallest integers which we can employ as direction-vectors?

6. Find the values of the direction-cosines of a line which satisfy the equation

$$\cos \alpha = 2 \cos \beta = 3 \cos \gamma,$$

and the least integers which can be used as direction-vectors.

7. Find the direction-cosines of lines joining the following pairs of points:

(*a*) From the origin to the point (2, 3, 4);
(*b*) From the origin to the point (− 2, − 3, − 4);
(*c*) From the point (1, 1, 1) to the point (2, 3, − 1);
(*d*) From the point (1, 2, − 3) to the point (− 1 − 3, 3).

If the order of the points of each pair be reversed, what effect will this change have on the direction-cosines?

8. What angle is formed by the two lines passing from the origin to the points (1, 1, 2) and (2, 3, 4) respectively?

9. Find the angle whose vertex is at the point (2, 3, 4), and whose sides pass through the points (1, 2, 3) and (3, 5, 5).

10. What angle is contained by two lines whose direction-vectors are:

Line A, $l = +1$; $m = +3$; $n = -5$;
Line B, $l = -3$; $m = +2$; $n = +1$.

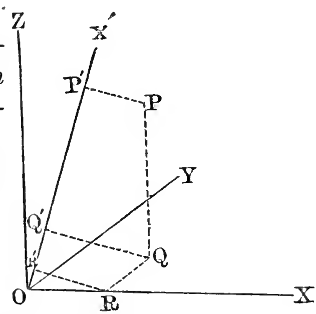
219. *Transformation of Co-ordinates.*

CASE I. *Transformation to a new system whose axes are parallel to those of the first system.* Let the co-ordinates of a point P referred to the old system be x, y and z, and let the co-ordinates of the new origin referred to the old system be a, b and c. It is now required to express the co-ordinates x', y', z' of P referred to the new system.

Because the new and old co-ordinate planes are parallel, the perpendiculars dropped from the point P upon corresponding planes will be coincident, and that portion of a perpendicular intercepted between the parallel planes will be a, b or c according as the plane is YZ, ZX or XY. The difference between the co-ordinates will therefore be equal to these same quantities, and we shall have

$$\left.\begin{array}{lll} x' = x - a; & y' = y - b; & z' = z - c; \\ \text{or} \quad x = x' + a; & y = y' + b; & z = z' + c. \end{array}\right\} \quad (11)$$

220. CASE II. *Transformation to a new rectangular system having the same origin but different directions.* Let O-XYZ be the axes of the old system, OX' any axis of the new system, and P a point whose co-ordinates are to be expressed in both systems.



From P drop $PQ \perp$ plane XY, $\therefore \parallel$ axis OZ.
From Q drop $QR \perp$ axis OX.

Then, calling x, y and z the co-ordinates of P referred to the old system, we shall have, by § 213,

$$x = OR;$$
$$y = RQ;$$
$$z = QP.$$

From the points P, Q and R drop perpendiculars upon the new axis OX', meeting it in the points P', Q' and R'. Let us then put

α, β, γ, the angles which OX', the axis of the new system, makes with the respective axes of X, Y and Z in the old system. We shall then have

$$OR' = OR \cos XOX' = x \cos \alpha;$$

also, because $RQ \parallel OY$,

$$R'Q' = RQ \cos YOX' = y \cos \beta;$$

also, because $QP \parallel OZ$,

$$Q'P' = QP \cos ZOX' = z \cos \gamma.$$

Now, the line OP' is the algebraical sum of the three segments OR', $+ R'Q'$, $+ Q'P'$, each segment being taken positively or negatively according as the angle α, β or γ is acute or obtuse.

Hence

$$OP' = x \cos \alpha + y \cos \beta + z \cos \gamma. \tag{12}$$

If we suppose OX' to represent the new axis of X, then OP' will be the co-ordinate x referred to the new axis, which we call x'. In the same way we have y' and z' when OX' represents the corresponding axes. If, therefore, we put

(X', X), (X', Y), (X', Z) the angles made by X' with X, Y & Z;
(Y', X), (Y', Y), (Y', Z) the angles made by Y' with X, Y & Z;
(Z', X), (Z', Y), (Z', Z) the angles made by Z' with X, Y & Z,

we shall have

$$\left.\begin{aligned} x' &= x \cos (X', X) + y \cos (X', Y) + z \cos (X', Z); \\ y' &= x \cos (Y', X) + y \cos (Y', Y) + z \cos (Y', Z); \\ z' &= x \cos (Z', X) + y \cos (Z', Y) + z \cos (Z', Z). \end{aligned}\right\} \quad (a)$$

The relation of the symbols (X', X), etc., to the symbols x, y, z and x', y', z', which is readily seen, renders these equations easy to write. But the subsequent management of the equations will be more simple if we retain the symbols α, β and γ, putting

$$\begin{array}{lll} \alpha, \ \beta \text{ and } \gamma & \text{for } (X', X), (X', Y) \text{ and } (X', Z); \\ \alpha', \ \beta' \text{ and } \gamma' & \text{for } (Y', X), (Y', Y) \text{ and } (Y', Z); \\ \alpha'', \ \beta'' \text{ and } \gamma'' & \text{for } (Z', X), (Z', Y) \text{ and } (Z', Z). \end{array}$$

The equations (a) will then be written

$$\left. \begin{aligned} x' &= x \cos \alpha + y \cos \beta + z \cos \gamma; \\ y' &= x \cos \alpha' + y \cos \beta' + z \cos \gamma'; \\ z' &= x \cos \alpha'' + y \cos \beta'' + z \cos \gamma''. \end{aligned} \right\} \qquad (13)$$

Each set of these cosines must separately satisfy the equation (7), which gives the first three equations written below. The last three are obtained by the consideration that, by § 216, the cosine of the angle between the axes of X' and Y' is

$$\cos \alpha \cos \alpha' + \cos \beta \cos \beta' + \cos \gamma \cos \gamma'.$$

But, because the new axes are rectangular, this cosine must be zero, as must also be the cosines of the angles between Y' and Z', and between Z' and X'. Thus we have the six equations of condition,

$$\left. \begin{aligned} &\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1; \\ &\cos^2\alpha' + \cos^2\beta' + \cos^2\gamma' = 1; \\ &\cos^2\alpha'' + \cos^2\beta'' + \cos^2\gamma'' = 1; \\ &\cos \alpha \cos \alpha' + \cos \beta \cos \beta' + \cos \gamma \cos \gamma' = 0; \\ &\cos \alpha' \cos \alpha'' + \cos \beta' \cos \beta'' + \cos \gamma' \cos \gamma'' = 0; \\ &\cos \alpha'' \cos \alpha + \cos \beta'' \cos \beta + \cos \gamma'' \cos \gamma = 0. \end{aligned} \right\} \qquad (14)$$

There being six separate equations of condition between the nine cosines, it follows that all nine of them can be expressed in terms of some three independent quantities. How this can be done we shall show hereafter.

221. We next remark that we can express the co-ordinates x, y and z in terms of x', y' and z', by reasoning exactly as we have reasoned in the reverse case, thus obtaining

$$\left.\begin{aligned} x &= x' \cos \alpha + y' \cos \alpha' + z' \cos \alpha''; \\ y &= x' \cos \beta + y' \cos \beta' + z' \cos \beta''; \\ z &= x' \cos \gamma + y' \cos \gamma' + z' \cos \gamma''. \end{aligned}\right\} \quad (15)$$

We can also derive the first of these equations directly from (12) by multiplying the first by cos α, the second by cos α' and the third by cos α'', and adding, noting the application of the results of § 216 to the angles formed by the axes.

Continuing the reasoning, we are led to the six equations of condition,

$$\left.\begin{aligned} &\cos^2\alpha + \cos^2\alpha' + \cos^2\alpha'' = 1; \\ &\cos^2\beta + \cos^2\beta' + \cos^2\beta'' = 1; \\ &\cos^2\gamma + \cos^2\gamma' + \cos^2\gamma'' = 1; \\ &\cos \alpha \cos \beta + \cos \alpha' \cos \beta' + \cos \alpha'' \cos \beta'' = 0; \\ &\cos \beta \cos \gamma + \cos \beta' \cos \gamma' + \cos \beta'' \cos \gamma'' = 0; \\ &\cos \gamma \cos \alpha + \cos \gamma' \cos \alpha' + \cos \gamma'' \cos \alpha'' = 0. \end{aligned}\right\} \quad (16)$$

In reality these equations are equivalent to the equations (14), and the one set can be deduced from the other by algebraic reasoning, without any reference to co-ordinates.

222. *Polar Co-ordinates in Space.* In space, as in a plane, the position of a point is determined when its *direction* and *distance* from the origin are given.

In space the direction requires two data to determine it. These data may be expressed in various ways, of which the following is the most common. We take, for positions of reference:

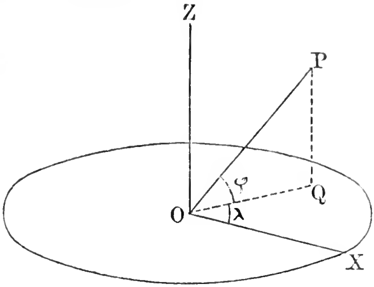
1. A fixed plane, called the *fundamental plane.* For this the plane of XY in rectangular co-ordinates is generally chosen.

2. An origin or pole, O, in this plane.

3. A line of reference, for which we commonly choose the axis of X.

Let P be the point whose position is to be defined. We first have to define the direction of the line OP. From any point P of this line drop a perpendicular PQ upon the fun-

damental plane, and join OQ. The direction of OP is then defined by the following two angles:



(1) The angle POQ which OP forms with its projection OQ; that is, the angle between OP and the plane.

(2) The angle XOQ which the projection of OP makes with OX.

It will be remarked that the planes of these two angles are perpendicular to each other.

To show that these two angles completely fix the direction of OP, we first remark that when the angle XOQ is given, the line OQ is fixed.

Next, because PQ is perpendicular to the plane, the point P and therefore the line OP must lie in the plane ZOQ, which is fixed because its two lines OZ and OQ are fixed. If the angle QOP in this (vertical) plane is given, there is only one line, OP, which can form this angle.

Hence the direction of the line OP is completely determined by the two angles XOQ and QOP; and when the distance OP is given, the point P is completely fixed.

We use the notation

φ, the angle QOP, or the elevation of OP above the plane XOY. We may call this angle the *latitude* of P.

λ, the angle XOQ which OQ, the projection of OP, makes with OX. We may call this angle the *longitude* of P.

r, the length of OP.

Because the quantities φ, λ and r completely fix the position of P, they are called the **polar co-ordinates** of P in space.

223. *Relation of the Preceding System to Latitude and Longitude.* For another conception of the angles φ and λ, pass a sphere around O as a centre, and mark on its surface the points and lines in which the lines and planes belonging to the preceding figure intersect it. Then

The fundamental plane OXQ intersects the spherical surface in the great circle XQY;

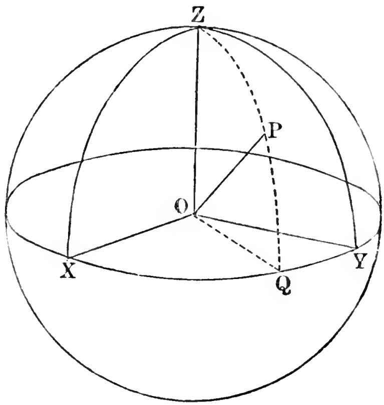
The line OX intersects it in X;

The line OQ intersects it in Q;

The lines OP and OZ intersect it in P and Z.

We therefore have

Angle XOQ *measured by* arc XQ;
Angle QOP *measured by* arc QP.



If now we imagine this sphere to be the earth, the great circle XY to be its equator, Z to be one of the poles, and P any point on its surface, then

The arc QP or the angle QOP is the latitude of P;

The arc XQ or angle XOQ is the longitude of P, counted from ZX as a prime meridian. Thus the angles we have been defining may be described under the familiar forms of longitude and latitude.

224. PROBLEM. *To transform the position of a point from rectangular to polar co-ordinates, and vice versa.*

Comparing the definitions of rectangular and polar co-ordinates, we put, for the point P in § 222,

$$PQ = z = OP \sin \varphi;$$
$$OQ = OP \cos \varphi.$$

Now, supposing a perpendicular dropped from Q upon OX, this perpendicular will be the ordinate y, and will meet OX at the distance x from the origin. Thus,

$$x = OQ \cos XOQ = OQ \cos \lambda;$$
$$y = OQ \sin XOQ = OQ \sin \lambda.$$

Putting $OP \equiv r$, and substituting for OQ its value, we have

$$\left.\begin{array}{l} x = r \cos \varphi \cos \lambda; \\ y = r \cos \varphi \sin \lambda; \\ z = r \sin \varphi; \end{array}\right\} \qquad (17)$$

which are the required equations.

Cor. The direction-cosines of the line OP in terms of φ and λ are

$$\left.\begin{array}{l} \cos \alpha = \cos \varphi \cos \lambda; \\ \cos \beta = \cos \varphi \sin \lambda; \\ \cos \gamma = \sin \varphi. \end{array}\right\} \qquad (18)$$

225. The result stated in § 220, that the nine direction-cosines of one system of rectangular axes with respect to another system can be expressed in terms of three independent quantities, may now be proved as follows:

1. Let OP be any one accented axis, say X'; the direction-cosines of this axis are expressed in terms of two angles, φ and λ, by (18).

2. Imagine a plane $\equiv M$, passing through O, § 223, perpendicular to OP. This plane M will be completely determined by the direction OP; whence the line $\equiv N$ in which it cuts the fundamental plane XY will also be determined.

3. The new axis Y' may lie in any direction from the point O in the plane M. One more angle $\equiv \psi$ is required to determine this direction, and for this angle we may take the angle which Y' forms with the line N.

4. The direction of the axis Z' is then completely fixed, because it must lie in the plane M and make an angle of 90° with Y'.

Thus, φ, λ and ψ completely determine the directions of the three new axes.

EXERCISES.

1. If, in the figure of § 223, the co-ordinates of the point P are

$$x = 27, \qquad y = 19, \qquad z = 17,$$

find its polar co-ordinates r, φ and λ.

Method of Solution. The quotient of the first two equations (17) gives

$$\tan \lambda = \frac{y}{x},$$

from which λ is found. Then we find $\sin \lambda$ or $\cos \lambda$ or both, and compute

$$r \cos \varphi = \frac{x}{\cos \lambda} = \frac{y}{\sin \lambda}.$$

Next we have

$$\tan \varphi = \frac{z}{r \cos \varphi},$$

from which we find φ. Then

$$r = \frac{z}{\sin \varphi} = \frac{r \cos \varphi}{\cos \varphi}.$$

2. Supposing the radius of the earth to be 6369 kilometres, the longitude of New York to be 74° west of Greenwich, and its latitude to be 40° 32′, it is required to find the rectangular co-ordinates of New York referred to the following system of axes having the earth's centre as the origin:

X in the equator, and on the meridian of Greenwich.
Y in the equator, in longitude 90° east of Greenwich.
Z passing through the North Pole.

3. If, in the figure of § 222, we take a point P' whose latitude is the same as that of P and whose longitude is 90° greater than that of P, it is required to express the angle POP'.

4. If the angle φ is negative, within what region will the point P be situated?

5. If we take a point P' whose latitude is φ and whose longitude is $\lambda + 180°$, how will it be situated relatively to P, and what will be the angle POP'?

6. If we take a point P' for which

$$\varphi' = 180° - \varphi,$$
$$\lambda' = \lambda + 180°,$$

show that this point will be identical with P.

CHAPTER II.

THE PLANE.

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226. *Introductory Considerations on the Loci of Equations.* If the values of the three co-ordinates of a point are not subject to any restriction, the point may occupy any position in space. Restrictions upon the position are algebraically expressed by equations of condition between the co-ordinates. Let us inquire what will be the locus of the point when the co-ordinates are required to satisfy a single equation of condition. By means of such an equation we may express any one of the co-ordinates, z for example, in terms of the other two, the form being

$$z = f(x, y). \qquad (a)$$

We can now assign any values we please to x and y, and for each pair of such values find the corresponding value of z. To each pair of values of x and y will correspond a certain point on the plane of XY. If at this point we erect a perpendicular equal to z, the end of each perpendicular will be a point whose co-ordinates satisfy the equation.

We may conceive these perpendiculars to become indefinitely numerous and indefinitely near each other, thus tending to form a solid. Their ends will then tend to form the surface of this solid. But these ends are the locus of the equation (a). Hence

The locus of a single equation of condition among the co-ordinates is a surface.

If a second equation is required to subsist among the co-ordinates, the locus of this equation will be a second surface.

If the co-ordinates are required to fulfil both conditions *simultaneously*, then the point must lie in *both* surfaces; that

is, it must lie on the line in which the surfaces intersect. Hence

The locus of two simultaneous equations between the co-ordinates is a line.

227. *To find the Equation of a Plane.* The property of a plane from which the locus can be most elegantly deduced is this: If on any line which intersects the plane perpendicularly we take two points, A and B, equidistant from the point of intersection and on opposite sides, then every point of the plane will be equidistant from A and B, and no point not on the plane will be equidistant.

Let us then drop from the origin a perpendicular upon the plane, and continue it to a distance on the other side equal to its length, and let P be the point at which it terminates. The condition that a point shall lie on the plane will then be that it shall be equidistant from the origin and from P.

Let us put

$a, b, c,$ the co-ordinates of P;

$x, y, z,$ the co-ordinates of any point on the plane.

Then, by §§ 214, 215, the squares of the distances of (x, y, z) from the origin and from P will be respectively

$$x^2 + y^2 + z^2$$

and $$(x - a)^2 + (y - b)^2 + (z - c)^2.$$

Developing and equating these two expressions, we find, for the required equation,

$$2ax + 2by + 2cz = a^2 + b^2 + c^2.$$

To reduce this equation, let us put

p, the length of the perpendicular dropped from the origin upon the plane; that is, one half the line from the origin to P;

$\alpha, \beta, \gamma,$ the angles which this perpendicular makes with the respective axes of X, Y and Z.

We shall then have $OP = 2p$, and the values of a, b and c will be, by § 214,

$$a = 2p \cos \alpha;$$
$$b = 2p \cos \beta;$$
$$c = 2p \cos \gamma.$$

Substituting these values in the equation of the plane, reducing, and remarking that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1,$$

the equation of the plane becomes

$$x \cos \alpha + y \cos \beta + z \cos \gamma - p = 0, \tag{1}$$

which is called the **normal equation** of the plane.

228. *The Angles α, β and γ.* As we have defined the angles α, β and γ, they are the angles which the perpendicular p forms with the co-ordinate axes. It is shown in Solid Geometry that the angle between any two planes is equal to that between any two lines perpendicular to them. Because

$$\text{Plane } YZ \perp \text{ axis } X,$$
$$\text{Plane (1) } \perp \text{ line } p,$$

$\therefore$ plane (1) makes the angle α with the plane YZ. Hence *we may define α, β and γ as the angles which the plane makes with the co-ordinate planes YZ, ZX and XY respectively.*

One restriction upon this proposition is necessary. Two supplementary angles are formed by any two planes, so that, in the absence of any convention, we should say that the angle between the planes is either equal or supplementary to that between the lines. The best way of avoiding ambiguity is to choose, for the angle between the planes, the angle between the perpendiculars dropped from the origin upon the planes.

If the angles α, β and γ are the same for several planes, these planes, being perpendicular to the same line, are parallel. Hence they may, in a certain sense, be said to have the same direction. We may therefore call $\cos \alpha$, $\cos \beta$ and $\cos \gamma$ the *direction-cosines* of the plane. They are also the direction-cosines of the perpendicular dropped from the origin upon the plane.

229. THEOREM I. *Every equation of the first degree between rectangular co-ordinates in space is the equation of some plane.*

Proof. Let the equation be

$$Lx + My + Nz + D = 0.$$

Divide this equation by $\sqrt{L^2 + M^2 + N^2}$, and determine three angles, α, β and γ, by the equations

$$\left.\begin{aligned} \cos\alpha &= \frac{L}{\sqrt{L^2 + M^2 + N^2}}; \\ \cos\beta &= \frac{M}{\sqrt{L^2 + M^2 + N^2}}; \\ \cos\gamma &= \frac{N}{\sqrt{L^2 + M^2 + N^2}}. \end{aligned}\right\} \qquad (2)$$

This will always be possible, because each of these cosines is less than unity. Because these cosines fulfil the condition (7), § 217, we can draw a line of length $= \dfrac{-D}{\sqrt{L^2+M^2+N^2}}$ from the origin making the angles α, β and γ with the several co-ordinate axes. Through the end of this line pass a plane perpendicular to it. Then, by the last section, the equation of this plane will be

$$x\cos\alpha + y\cos\beta + z\cos\gamma + \frac{D}{\sqrt{L^2 + M^2 + N^2}} = 0,$$

an equation which becomes identical with that assumed in the hypothesis by clearing of denominators and substituting from (2). We may therefore put the theorem in the following more specific form:

Every equation of the form

$$Lx + My + Nz + D = 0$$

represents a certain definite plane, namely, the plane passing perpendicularly through the end of that line which

emanates from the origin;

makes with the axes angles whose cosines are

$$\frac{L}{\sqrt{L^2 + M^2 + N^2}}, \quad \frac{M}{\sqrt{L^2 + M^2 + N^2}} \quad \textit{and} \quad \frac{N}{\sqrt{L^2 + M^2 + N^2}}$$

respectively;

and has the length $\dfrac{D}{\sqrt{L^2 + M^2 + N^2}}$.

230. *Notation.* By "the plane (L, M, N, D)" we mean "the plane whose equation is

$$Lx + My + Nz + D = 0."$$

Def. An equation of a plane in which the four quantities L, M, N and D are all independent is called the **general equation** of a plane.

The general equation may be considered as related in two ways to the normal or other special forms of the equation.

I. The special forms are cases in which certain relations exist among the quantities L, M, N and D. For example, the normal form is the special case in which $L^2 + M^2 + N^2 = 1$.

Whenever we find this condition satisfied, we know that the equation is in the normal form.

II. The general equation may always be reduced to the normal form by dividing by $\sqrt{L^2 + M^2 + N^2}$.

Direction-Vectors. From the equation (2) it is seen that L, M and N may be taken as the direction-vectors of any line perpendicular to the plane, because they are severally equal to the direction-cosines of such a line multiplied by the common factor $\sqrt{(L^2 + M^2 + N^2)}$. Hence we conclude:

The equation of every plane perpendicular to a line whose direction-vectors are l, m and n may be written in the form

$$lx + my + nz + d = 0;$$

and, conversely, *for the direction-vectors of any line perpendicular to the plane (L, M, N, D) may be taken L, M and N.*

231. *Special Positions of a Plane.* If one of the coefficients L, M, or N vanishes, the cosine of the angle which the plane makes with the corresponding co-ordinate plane will also vanish; that is, the plane will be perpendicular to the co-ordinate plane, and therefore parallel to the axis of that plane. Hence *an equation of the first degree between two only of the co-ordinates represents a plane parallel to the axis of the missing co-ordinate.*

For example, the locus of

$$Lx + My + D = 0$$

is a plane parallel to the axis of Z.

It follows that if two co-ordinates are missing, the locus will be parallel to the common plane of the missing co-ordinates. For example, the locus of

$$Lx + D = 0$$

or

$$x = -\frac{D}{L}$$

will be a plane perpendicular to the axis of X and parallel to the plane YZ.

232. *Lines and Points connected with a Plane.* The following lines and points are determined by every plane:

I. The three lines in which it intersects the co-ordinate planes.

II. The three points in which it intersects the co-ordinate axes.

III. The foot of the perpendicular from the origin upon the plane.

When we include among possible lines and points the lines and points at infinity, the above three lines and four points will always be determinate.

Def. The lines in which a plane intersects the co-ordinate planes are called **traces** of the plane.

The distances from the origin to the three points in which a plane cuts the co-ordinate axes are called the **intercepts** of the axes by the plane.

233. PROBLEM. *To find the equations of the traces of a plane.*

The trace of the plane upon the plane of YZ is simply those points of the plane for which $x = 0$. Hence, if we put $x = 0$ in the equation of a plane, we have the equation of its trace upon YZ. Therefore, in the general equation

$$Lx + My + Nz + D = 0,$$

the equations of the traces upon the co-ordinate planes are:

$$\begin{array}{ll} \text{On } YZ, & My + Nz + D = 0; \\ \text{On } ZX, & Lx + Nz + D = 0; \\ \text{On } XY, & Lx + My + D = 0. \end{array}$$

These equations, representing lines upon planes, can be discussed like the equations of lines in Plane Analytic Geometry.

234. PROBLEM. *To express the lengths of the intercepts of the axes by a plane.*

At the point where the plane cuts the axis of X we have $y = 0$ and $z = 0$. Hence the intercept is the value of x corresponding to zero values of y and z, and so with the other co-ordinates. Thus:

$$\left.\begin{aligned} \text{Intercept on } X &= -\frac{D}{L}; \\ \text{Intercept on } Y &= -\frac{D}{M}; \\ \text{Intercept on } Z &= -\frac{D}{N}. \end{aligned}\right\} \qquad (4)$$

Scholium. Each of the traces necessarily meets the other two on the several co-ordinate axes, and their points of meeting are those in which the plane cuts the axes. Hence the traces form a plane triangle of which the points in which the plane intercepts the axes are the vertices.

Each of the sides of this triangle is the hypothenuse of a right triangle of which the sides containing the right angle are the intercepts upon the axes.

The relations between the sides and angles of these triangles, considered individually, may be investigated by the methods of Plane Trigonometry.

235. PROBLEM. *To express the equation of a plane in terms of its intercepts upon the axes*

Let us put

a, b, c, the intercepts on the axes of X, Y and Z respectively.

Then

$$\left.\begin{aligned} a &= -\frac{D}{L}, \quad \therefore \quad L = -\frac{D}{a}; \\ b &= -\frac{D}{M}, \quad \therefore \quad M = -\frac{D}{b}; \\ c &= -\frac{D}{N}, \quad \therefore \quad N = -\frac{D}{c}. \end{aligned}\right\} \qquad (5)$$

Substituting these values of L, M and N in the general equation, it reduces to

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1, \tag{6}$$

which is the required equation.

EXERCISES.

1. Write the equation of that plane for which the co-ordinates of the foot of the perpendicular from the origin upon the plane are 1, 2, and 3.

2. If, in the general equation of a plane, the coefficients L, M and N are all equal, what angle will the perpendicular make with the co-ordinate axes, and what angle will the plane make with the co-ordinate planes?

3. The equation of a plane being

$$3x + 4y - 12z = 26,$$

it is required to reduce it to the normal form to find the angles which it forms with the co-ordinate axes, the equations of its traces upon the co-ordinate planes, the lengths of its intercepts upon the co-ordinate axes, the lengths of its traces between these intercepts, and its least distance from the origin.

4. The intercepts being in the proportion $1 : 2 : 3$, what are the cosines of the angles which the perpendicular upon the plane makes with the axes?

5. Show that the inverse square of the perpendicular from the origin upon a plane is equal to the sum of the inverse squares of its intercepts; i.e.,

$$\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}.$$

6. Show the corresponding relation between the two sides of a right triangle and the perpendicular from the vertex upon the hypothenuse.

7. Express the lengths of the sides of the triangle formed by the traces of a plane in terms of the intercepts, and prove that the sum of the squares of the sides is twice the sum of the squares of the intercepts.

8. A plane cuts traces whose lengths between the axes are:
On plane YZ, u; on plane ZX, v; on plane XY, w.
Find the lengths of the intercepts and the equation of the plane in terms of $s^2 = \frac{1}{2}(u^2 + v^2 + w^2)$.

$$\textit{Ans.}\quad \frac{x}{\sqrt{s^2 - u^2}} + \frac{y}{\sqrt{s^2 - v^2}} + \frac{z}{\sqrt{s^2 - w^2}} = 1.$$

9. Find the angle included between the planes

$$x + y + z = a$$

and

$$x - y + 2z = b. \quad \text{(Comp. §§ 216, 230)}$$

236. *Plane satisfying Given Conditions.* If a plane is required to satisfy a condition, that condition can be expressed as an equation between the constants L, M, N, D, which determine the position of the plane. By means of this equation one of the constants can be eliminated from the equation of the plane, and the condition will then be fulfilled for all values of the remaining constants.

If two conditions are given, two constants can be eliminated; if three, all the constants. For, although the general equation of the plane contains four constants, it depends only on the three ratios of any three of these constants to the fourth. In fact, we can always reduce the general equation to the form (6)

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1,$$

which contains but three arbitrary constants.

237. PROBLEM. *To find the equation of a plane passing through a given point.*

Let (x', y', z') be the given point. In order that the plane (L, M, N, D) may pass through this point, its constants must satisfy the condition

$$Lx' + My' + Nz' + D = 0, \tag{a}$$

which gives

$$D = -(Lx' + My' + Nz').$$

Substituting this value of D in the general equation, the latter becomes

$$Lx + My + Nz - (Lx' + My' + Nz') = 0, \tag{7}$$

or, in another form,

$$L(x - x') + M(y - y') + N(z - z') = 0. \tag{8}$$

REMARK 1. In these equations we may assign any values we please to L, M and N, without the plane ceasing to pass through the point (x', y', z'), as is evident from (8).

REMARK 2. If we had two equations of the form (a), we could eliminate two of the constants, say N and D, and L and M would still remain. If we had three equations, we could eliminate three constants, M, N and D for example. That is, we could, by solving the equations, express M, N and D in terms of L. Substituting these values in the general equation, the latter would, it would seem, still contain the constant L. But, in reality, L would enter only as a factor of the whole equation, and would therefore divide out. Hence, when we eliminate any three of the four constants, the fourth drops out of itself without the introduction of any further condition.

Relations of Two or More Planes.

238. *Parallel and Perpendicular Planes.*

THEOREM II. *If, in the equations of any two planes*

$$Lx + My + Nz + D = 0,$$
$$L'x + M'y + N'z + D' = 0,$$

the direction-vectors L', M' and N' are proportional to L, M and N respectively, the two planes are parallel.

Proof. The direction-vectors of the perpendiculars from the origin upon the planes being proportional, the direction-cosines are equal (§ 217), and these perpendiculars are coincident (§ 218). Hence the planes are perpendicular to the same line and therefore parallel. Q. E. D.

PROBLEM. *To find the condition that two planes given by their equations shall be perpendicular to each other.*

Let the planes be (L, M, N, D) and (L', M', N', D').

The planes will be perpendicular when the perpendiculars from the origin are perpendicular to each other. The condition is, from eq. (9) of § 217,

$$LL' + MM' + NN' = 0.$$

239. *Notation.* I. We use the symbols P, P', P'', etc., Q, Q', etc. etc., to signify functions of the co-ordinates of the first degree. For example,

$$P \equiv Lx + My + Nz + D;$$
$$P' \equiv L'x + M'y + N'z + D';$$
$$\text{etc.} \qquad \text{etc.} \qquad \text{etc.}$$

II. By the expression "the plane P" we mean the plane whose equation is $P = 0$.

240. THEOREM III. *If in a function P we substitute for x, y and z the co-ordinates x', y' and z' of a point, P will then express the distance of that point from the plane $P = 0$ multiplied by the factor $\sqrt{L^2 + M^2 + N^2}$.*

Proof. Let us pass through the point (x', y', z') a plane A parallel to the plane P. The equation of this plane will be (§ 237)

$$Lx + My + Nz - (Lx' + My' + Nz') = 0.$$

The term independent of x, y and z is $-(Lx'+My'+Nz')$, which takes the place of D in the general equation. Hence the perpendicular distance of this plane A from the origin is

$$p' = \frac{Lx' + My' + Nz'}{\sqrt{L^2 + M^2 + N^2}}.$$

The perpendicular distance of the plane P from the origin is (§ 229)

$$p = \frac{-D}{\sqrt{L^2 + M^2 + N^2}}.$$

Because the point (x', y', z') is in the plane $A \parallel P$, the distance of (x', y', z') from the plane P is equal to the constant distance between the planes, and hence to the difference $p - p'$ between the perpendiculars. Hence

$$\text{Distance of } (x', y', z') \text{ from } P = \frac{Lx' + My' + Nz' + D}{\sqrt{L^2 + M^2 + N^2}},$$

or

$$Lx' + My' + Nz' + D = \text{Distance} \times \sqrt{L^2 + M^2 + N^2}.$$

Q. E. D.

Cor. If the equation is in the normal form, we shall have

$$\sqrt{L^2 + M^2 + N^2} = 1,$$

and the expression P will then represent the distance of the point whose co-ordinates appear in it from the plane $P = 0$.

EXERCISE.

Show that the angle ε between the two planes

$$Lx + My + Nz + D = 0 \quad \text{and} \quad L'x + M'y + N'z + D' = 0$$

is given by the equation

$$\cos \varepsilon = \frac{LL' + MM' + NN'}{\sqrt{L^2 + M^2 + N^2}\sqrt{L'^2 + M'^2 + N'^2}}.$$

241. THEOREM IV. *If $P = 0$ and $P' = 0$ be the equations of any two planes, and λ and λ' constants, the equation*

$$\lambda P + \lambda' P' = 0 \qquad (a)$$

will be the equation of a third plane intersecting the other two in the same line.

Proof. I. The expression $\lambda P + \lambda' P'$ is of the first degree in x, y and z. Therefore the equation is that of some plane.

II. Every set of values of the co-ordinates x, y and z which simultaneously satisfy both equations $P = 0$ and $P' = 0$ also satisfy equation (a). The co-ordinates which satisfy both equations are those of their line of intersection (§ 226). Therefore these co-ordinates also satisfy (a); whence the line lies in the plane $\lambda P + \lambda' P'$, which proves the theorem.

Cor. If three functions, P, P' and P'', are such that it is possible to find three constant coefficients, λ, λ' and λ'', which lead to the identity

$$\lambda P + \lambda' P' + \lambda'' P'' \equiv 0,$$

the three planes P, P' and P'' intersect in the same line.

THEOREM V. *If $Q = 0$ and $Q' = 0$ are the equations of two planes in the normal form, the equations*

$$Q + Q' = 0 \qquad \textit{and} \qquad Q - Q' = 0$$

represent the planes which bisect the dihedral angles formed by the plane Q and Q'.

Proof. Because the expressions Q and Q' represent the distances of the point (x, y, z) from the planes Q and Q' (§ 240), it follows that the equation

$$Q = Q' \qquad \text{or} \qquad Q - Q' = 0$$

will express the condition that the said point is equally distant from the planes Q and Q'. Hence it lies upon the plane bisecting the angle formed by Q and Q', and this plane is the locus of the equation $Q - Q' = 0$.

The equation $Q + Q' = 0$ is equivalent to $Q = -Q'$, and asserts that the point (x, y, z), if on the positive side of the one plane, is on the negative side of the other at an equal distance. Therefore it bisects the adjacent dihedral angle.

Cor. In the case supposed, the two planes $Q - Q'$ and $Q + Q'$ are perpendicular to each other because they are the bisectors of adjacent angles.

Theorem VI. *If λ, λ' and λ'' are constant coefficients, the equation*

$$\lambda P + \lambda' P' + \lambda'' P'' = 0 \qquad (b)$$

represents a plane passing through the common point of intersection of the planes P, P' and P''.

Proof. In the same way as with Theorem IV, it is shown (1) that the equation is that of a plane, and (2) that the co-ordinates of any and every point common to the three planes P, P' and P'' satisfy equation (b). Now, because any three planes have one point common (which may be at infinity), the point common to P, P' and P'' lies on the plane (b). Q. E. D.

Cor. If four functions, P, P', P'' and P''', are such that an identity of the form

$$\lambda P + \lambda' P' + \lambda'' P'' + \lambda''' P''' \equiv 0$$

is possible, the four planes P, P', P'' and P''' will intersect in a point.

242. *Bisectors of Dihedral Angles.* The foregoing principles enable us to prove many elegant relations among the planes which bisect the dihedral angles of a solid, or of a solid angle. Let

$$Q = 0, \qquad Q' = 0, \qquad Q'' = 0,$$

be the equations of any three planes in the normal form. Since any three planes meet in a point, they may be considered as forming a dihedral angle at that point.

The bisecting planes of the three dihedral angles formed by the planes Q, Q' and Q'' will be (§ 241)

$$\begin{aligned} Q\ \ - Q'\ &= 0 \equiv P; \\ Q'\ - Q'' &= 0 \equiv P'; \\ Q'' - Q\ \ &= 0 \equiv P''. \end{aligned}$$

These functions satisfy the condition

$$P + P' + P'' \equiv 0.$$

Therefore *the three bisecting planes of a dihedral angle intersect on a line.*

Placing the centre of a sphere at the vertex of the dihedral angle, and considering the spherical triangle formed by the planes Q, Q', Q'', we have the theorem:

The great circles bisecting the interior angles of a spherical triangle meet in a point.

EXERCISES.

1. In order that the two planes $P+P'=0$ and $P-P'=0$ may be perpendicular to each other, show that the coefficients of x, y and z in P and P' must satisfy the condition

$$L^2 + M^2 + N^2 = L'^2 + M'^2 + N'^2.$$

2. Describe the relative position of the four planes

$$\begin{aligned} x + \ \ y + \ \ z &= 0, \\ x + \ \ y - 2z &= 0, \\ x - 2y + \ \ z &= 0, \\ -2x + \ \ y + \ \ z &= 0, \end{aligned}$$

and find the angles which each makes with the three others.

3. Show that the line of intersection of the two planes

$$ax + by + cz + d = 0,$$
$$ax + by - cz + d = 0,$$

is in the plane of XY, and that its equation in this plane is

$$ax + by + d = 0.$$

4. What is the condition that a plane shall pass through the origin?

5. Write the equation of a plane making equal angles with the three co-ordinate planes and cutting off from the axis of X an intercept a.

6. When a plane makes equal angles with the three co-ordinate planes, what is the ratio of each intercept which it cuts off from the axes to the perpendicular from the origin upon the plane? *Ans.* $\sqrt{3} : 1$.

7. Write the equation of a plane which shall make equal angles with the axes of X and Z, and shall be parallel to the axis of Y.

8. What is the distance apart of the parallel planes

$$x + 2y + 2z = a;$$
$$2x + 4y + 4z = b?$$

9. Write the equation of the plane which shall pass through the point (1, 2, 2) and be parallel to the plane

$$-x + 2y - z = 0.$$

10. Write the equation of the plane which shall pass through the origin, the point (1, 1, 2) and the point (2, 3, 1). *Ans.* $-5x + 3y + z = 0$.

11. Write the equation of the plane which shall pass through the origin and the point (1, 2, 2), and shall be perpendicular to the plane

$$x - y + z = 0.$$

Ans. $4x + y - 3z = 0$.

12. Find the locus of that point which is required to be equally distant from the points (a, b, c) and (a', b', c').

Ans. $2(a' - a)x + 2(b' - b)y + 2(c' - c)z$
$$= a'^2 - a^2 + b'^2 - b^2 + c'^2 - c^2.$$

13. If, in the preceding problem, the point (a', b', c') is on the straight line from the origin to (a, b, c), and m times as far from the origin as (a, b, c), show that the perpendicular from the origin upon the plane is $\dfrac{m+1}{2}\sqrt{a^2+b^2+c^2}$.

14. The plane $x + y + z - d = 0$ is required to bisect the line from the origin to the point (a, b, c). Find the value of d. *Ans.* $d = \frac{1}{2}(a + b + c)$.

15. Find the equation of the plane passing through the origin and through the line of intersection of the planes

$$2x + 3y + 4z + p = 0;$$
$$x + y + z - 2p = 0.$$

Ans. $5x + 7y + 9z = 0$.

16. Find the equation of the plane which shall pass through the point (2, 3, 5) and through the line of intersection of the two planes

$$x + y + z - 5 = 0;$$
$$x - y + 2z + 1 = 0.$$

Ans. $x + 3y - 11 = 0$.

Calling the two expressions P and P', the equation of any plane passing through the intersection of P and P' may be written in the form $\lambda P + P' = 0$. We determine λ by the condition that this equation shall be satisfied when we have $x = 2$, $y = 3$ and $z = 5$.

17. Write the equation of the plane passing through the origin and perpendicular to the two planes

$$x + y - z = 0;$$
$$x - y - 2z = 0.$$

Ans. $3x - y + 2z = 0$.

18. The three planes

$$x - 2y - 3z = 0,$$
$$2x + y - nz = 0,$$
$$l'x + m'y + n'z = 0,$$

are each to be perpendicular to the other two. Find the least integral values of l', m', n' and n which satisfy this condition, and thus show that the equations of the second and third planes are

$$2x + y = 0;$$
$$3x - 6y + 5z = 0.$$

CHAPTER III.

THE STRAIGHT LINE IN SPACE.

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243. THEOREM I. *The position of a line is completely determined by its projections upon any two non-parallel planes.*

Proof. Through the projection on one of the planes pass a plane $R \perp$ to that of projection. The line projected then must lie entirely in the plane R.

In the same way, the line must lie entirely in the plane $S \perp$ to the other plane of projection and containing the other projection. Hence the line is the intersection of the planes R and S.

There can be only one plane R and one plane S, because along a given line in a plane only one $\perp$ plane can be passed. Hence there is but one line in which these planes can intersect, and this is the line whose projections are given. Q. E. D.

244. *Equations of a Straight Line.* Since any one equation between the co-ordinates of a point represents a surface, at least two equations are necessary to represent a line in space. These equations, considered *separately,* represent two surfaces. Considered *simultaneously,* that is, requiring the co-ordinates to satisfy them both, they represent the line in which the surfaces intersect.

The most simple form of the equations of a straight line are given by the equations of the planes in which it is projected upon any two of the co-ordinate planes, XZ and YZ for example. The equation

$$x = hz + a$$

(y being left indeterminate) represents a certain plane parallel to the axis of Y (§ 231); that is, the co-ordinates of all the

points in this plane satisfy the equation, and *vice versa*. In the same way, every point whose co-ordinates satisfy the equation

$$y = kz + b$$

lies in a certain plane parallel to the axis of X. Hence every point whose co-ordinates satisfy *both* equations must lie in both planes, that is, in the line of intersection of the planes. The two equations taken simultaneously therefore represent a straight line.

REMARK. Any two consistent and independent simultaneous equations between the co-ordinates, for instance,

$$\left.\begin{aligned} ax + by + cz + d &= 0, \\ a'x + b'y + c'z + d' &= 0, \end{aligned}\right\} \qquad (1)$$

equally represent a straight line, namely, the line in which the planes intersect. But the forms

$$\left.\begin{aligned} x &= hz + a, \\ y &= kz + b, \end{aligned}\right\} \qquad (2)$$

are preferred because they are more simple.

We also remark that the form (1) can always be reduced to the form (2) by first eliminating y and then x from the two equations.

EXERCISES.

1. Express the equations of the line of intersection of the planes

$$\begin{aligned} 3x - 2y + z + 5d &= 0, \\ -x + y + 2z - 4d &= 0, \end{aligned}$$

in the form (2).

Ans. $\begin{cases} x = -5z + 3d; \\ y = -7z + 7d. \end{cases}$

2. Express in the form (2) the equations of the three lines of intersection of the planes

$$\begin{aligned} x - y - z &= a; \\ x + y - z &= b; \\ x + 2y + z &= a + b. \end{aligned}$$

3. Explain how it is that the equation of a line in one of the co-ordinate planes (the other co-ordinate being supposed zero) is the same as the equation of the plane passing through that line and parallel to the third co-ordinate.

4. Prove that if we represent the equations of a straight line [(1) or (2), for example] in the form

$$P = 0, \qquad Q = 0,$$

then the equations

$$mP + nQ = 0, \qquad mP - nQ = 0,$$

m and n being constants, will represent the same line.

245. *Symmetrical Equations of a Straight Line.* The equations of a straight line may be represented, not only by two equations between the three co-ordinates, but by expressing each of the three co-ordinates as a function of a fourth variable. To do this, let us put

x_0, y_0, z_0, the co-ordinates of any fixed point of the line;
x, y, z, the co-ordinates of any other point of the line;
ρ, the length between the points (x_0, y_0, z_0) and (x, y, z).

Then, α, β and γ being the angles which the line makes with the co-ordinate axes, we have, by § 215,

$$\left.\begin{aligned} x - x_0 &= \rho \cos \alpha; \\ y - y_0 &= \rho \cos \beta; \\ z - z_0 &= \rho \cos \gamma. \end{aligned}\right\} \qquad (3)$$

Here x_0, y_0, z_0, α, β and γ are supposed to be constants which determine the position of the line in space, while x, y, z and ρ are variables. Assigning any value we please to ρ, we shall have corresponding values of x, y and z, which will be the co-ordinates of that point P on the line which is at the distance ρ from the point (x_0, y_0, z_0). Since for every point on the line there will be one and only one value of ρ, and for this value of ρ one value and no more of each co-ordinate, and *vice versa*, the equations (3) will represent all points of the line, and no others. They are therefore the equations of a straight line.

These equations (3) are readily reduced to the form (2) by

eliminating ρ, first between the first and third, and then between the second and third. We thus find

$$x = \frac{\cos\alpha}{\cos\gamma}z + \frac{x_0\cos\gamma - z_0\cos\alpha}{\cos\gamma};$$
$$y = \frac{\cos\beta}{\cos\gamma}z + \frac{y_0\cos\gamma - z_0\cos\beta}{\cos\gamma}.$$

The equations may also be reduced to the symmetric form

$$\frac{x - x_0}{\cos\alpha} = \frac{y - y_0}{\cos\beta} = \frac{z - z_0}{\cos\gamma}.$$

246. *Introduction of Direction-Vectors.* In the equations (3) we may introduce, instead of the direction-cosines, any three quantities proportional to them, without changing the line represented by the equation. Let these quantities be l, m and n, so that the equations become

$$\left.\begin{aligned} x &= x_0 + l\rho; \\ y &= y_0 + m\rho; \\ z &= z_0 + n\rho. \end{aligned}\right\} \qquad (4)$$

To show that the line is unchanged, we proceed as in § 217, where we have shown that the proportionality of l, m and n to the direction-cosines may be expressed by the equations

$$= \sigma\cos\alpha; \qquad m = \sigma\cos\beta; \qquad n = \sigma\cos\gamma.$$

By substitution the equations (4) become

$$x = x_0 + \rho\sigma\cos\alpha;$$
$$y = y_0 + \rho\sigma\cos\beta;$$
$$z = z_0 + \rho\sigma\cos\gamma.$$

These equations are the same as (3), except that $\rho\sigma$ takes the place of ρ; that is, the distance between the points (x_0, y_0, z_0) and (x, y, z) is $\rho\sigma$ instead of ρ. Hence the systems (3) and (4) represent the same line, except that in (4) ρ represents length $\div\ \sigma$, instead of length simply.

Cor. We may multiply the three direction-vectors in the symmetrical equations of a line by any common factor without changing the line represented.

Example 5, page 265.

Assume equations (4) as the equa[tions] of the line passing thro' the point $(x_0 y_0 z_0)$:

$$\begin{aligned} x &= x_0 + l\rho_0 \qquad [l^2+m^2+n^2=1 \\ y &= y_0 + m\rho_0 \qquad\qquad (4) \\ z &= z_0 + n\rho_0. \quad [\rho_0 \text{ variable}] \end{aligned}$$

If $(x_1 y_1 z_1)$ lies also on the line, then

$$\begin{aligned} x_1 - x_0 &= l\rho_1 \\ y_1 - y_0 &= m\rho_1 \\ z_1 - z_0 &= n\rho_1, \quad [\rho_0 \text{ changes to } \rho_1] \end{aligned}$$

where ρ_1 is the distance between the points $(x_0 y_0 z_0)$, $(x_1 y_1 z_1)$. These equations determine l, m, n in terms of $(x_0 y_0 z_0)$, $(x_1 y_1 z_1)$ and ρ_1, which values substituted in (4) give

$$\begin{aligned} x &= x_0 + (x_1 - x_0)\frac{\rho_0}{\rho_1} \\ y &= y_0 + (y_1 - y_0)\frac{\rho_0}{\rho_1} \\ z &= z_0 + (z_1 - z_0)\frac{\rho_0}{\rho_1}, \end{aligned}$$

where $\rho_1 = \sqrt{(x_1-x_0)^2+(y_1-y_0)^2+(z_1-z_0)^2}$.

Now $\frac{x_1-x_0}{\rho_1}$, $\frac{y_1-y_0}{\rho_1}$, $\frac{z_1-z_0}{\rho_1}$ are the direction cosines of the line joining $(x_0 y_0 z_0)$ and $(x_1 y_1 z_1)$. Hence the equations are in the form:

$$\begin{aligned} x &= x_0 + \rho_0 \cos\alpha & &= x_0 + \rho\rho_1 \cos\alpha \\ y &= y_0 + \rho_0 \cos\beta \quad \text{or} & &= y_0 + \rho\rho_1 \cos\beta \\ z &= z_0 + \rho_0 \cos\gamma & &= z_0 + \rho\rho_1 \cos\gamma \end{aligned}$$

REMARK. The forms (3) and (4) have a great advantage in nearly all the investigations of Analytic Geometry, and will therefore be exclusively employed. The advantage arises from the fact that the three co-ordinates which fix the position of some one point of the line are completely distinct from the quantities l, m and n which express its direction.

EXERCISES.

1. Express the co-ordinates of the three points in which the line given by the equations (3) intersects the three co-ordinate planes respectively. Express also the corresponding values of ρ.

2. Write, in the form (4), the equations of a line passing through the point (a, b, c) and parallel to the axis of Z.

3. Write, in the same form, the equations of a line passing through the point (x_0, y_0, z_0) parallel to the plane of XY and making equal angles with the axes of X and Y.

4. Write the equations of a line passing through the point (x_0, y_0, z_0) and making equal angles with the co-ordinate planes. Express also the co-ordinates of the three points in which it intersects the co-ordinate planes.

Ans., in part. It intersects the plane of YZ in the points

$$y = y_0 - x_0; \qquad z = z_0 - x_0.$$

5. Show that the equations of the line passing through the points (x_0, y_0, z_0) and (x_1, y_1, z_1) may be written in the form

$$x = x_0 + (x_1 - x_0)\rho;$$
$$y = y_0 + (y_1 - y_0)\rho;$$
$$z = z_0 + (z_1 - z_0)\rho.$$

ρ is here equal to $\frac{\rho_0}{\rho_1}$ whe
$\rho_0 =$ dist. bet. $(x \ldots)$ and $(x$
$\rho_1 =$ " " $(x_0 \ldots)$ and $($

State to what distance on the line corresponds the unit of ρ in these equations, and find the co-ordinates of the points in which the line intersects the co-ordinate planes.

6. Write the three symmetrical equations of the straight line joining the points (1, 1, 2) and (2, 3, 5). Find the angles which it makes with the co-ordinate axes, the points in which it intersects the co-ordinate planes, and the distances between these points.

247. *Condition that a Line shall be parallel to a Plane.* So long as the coefficients l, m and n in the equations (4) of a straight line are entirely unrestricted, these equations may, by giving suitable values to l, m and n, be made to represent any line whatever passing through the point (x_0, y_0, z_0). If, however, they be subjected to a homogeneous equation of condition, the lines will be restricted, as we shall now show.

THEOREM II. *If, in the symmetrical equations of a straight line, the direction-vectors m, n and p are required to satisfy a linear equation, the line will lie in or be parallel to a certain plane.*

Conversely, *the requirement that the line shall lie in or be parallel to a certain plane is indicated by a linear equation between the direction-vectors.*

Proof. Let the linear equation which m, n and p are required to satisfy be

$$Al + Bm + Cn = 0. \qquad (a)$$

I say that every point of every possible line represented by the equations (4) will then lie in the plane whose equation is

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0, \qquad (b)$$

and will therefore be parallel to every plane whose direction-vectors are A, B and C. For, by multiplying the equations (4) respectively by A, B and C, transposing, and adding the products, we find

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = (Al + Bm + Cn)\rho.$$

Now, by hypothesis (a), the second member of this equation vanishes. Hence all values of the co-ordinates x, y and z which satisfy (4) also satisfy (b). Hence every point of the line lies in the surface whose equation is (b), and this surface is a plane, by § 229.

Every plane whose direction-vectors are A, B and C is parallel to (b), because perpendicular to the same line. Hence (a) is the condition that the line (4) is parallel to every such plane.

Next, let it be required that the line (5) shall lie in the plane whose equation is

$$Ax + By + Cz + D = 0. \qquad (c)$$

I say that the coefficients m, n and ~~p~~ l must satisfy the linear equation

$$Al + Bm + Cn = 0.$$

For, by substituting in (c) the values of x, y and z from (4), we have

$$Ax_0 + By_0 + Cz_0 + D + (Al + Bm + Cn)\rho = 0, \qquad (d)$$

which equation must be satisfied for all values of ρ. Now, by hypothesis, the point (x_0, y_0, z_0) lies on the line, and therefore lies in the plane (c) which requires it to satisfy the equation

$$Ax_0 + By_0 + Cz_0 + D = 0.$$

Hence, in order that the equation (d) may be satisfied, we must have

$$Al + Bm + Cn = 0. \qquad (5)$$

248. *Common Perpendicular to Two Lines.* It is shown in Geometry that two non-parallel lines have one and only one common perpendicular, and that this perpendicular is the shortest distance between the lines. Let us now solve the problem,

To find the equation of the common perpendicular to two given lines.

We shall express the equations of the given lines in the form (3), putting, for brevity,

$$\left.\begin{matrix}\alpha_1, \beta_1, \gamma_1, \\ \alpha_2, \beta_2, \gamma_2,\end{matrix}\right\} \text{the direction-cosines of the given lines,}$$

and

α, β, γ, those of the required perpendicular.

Thus the symmetrical equations of the given lines will be

$$\left.\begin{aligned} x &= x_1 + \alpha_1\rho; \\ y &= y_1 + \beta_1\rho; \\ z &= z_1 + \gamma_1\rho; \end{aligned}\right\} \quad \text{and} \quad \left\{\begin{aligned} x &= x_2 + \alpha_2\rho; \\ y &= y_2 + \beta_2\rho; \\ z &= z_2 + \gamma_2\rho. \end{aligned}\right.$$

Let us first find the direction-cosines α, β, γ. By §§ 216, 217, we have the equations

$$\left.\begin{array}{l}\alpha_1\alpha + \beta_1\beta + \gamma_1\gamma = 0;\\ \alpha_2\alpha + \beta_2\beta + \gamma_2\gamma = 0;\\ \alpha^2 \ \ + \beta^2 \ \ + \gamma^2 \ \ = 1.\end{array}\right\} \qquad (6)$$

Eliminating first β and then γ from these equations, we have

$$(\alpha_1\beta_2 - \alpha_2\beta_1)\alpha + (\beta_2\gamma_1 - \beta_1\gamma_2)\gamma = 0;$$
$$(\gamma_1\alpha_2 - \gamma_2\alpha_1)\alpha + (\beta_2\gamma_1 - \beta_1\gamma_2)\beta = 0.$$

Dividing these equations by $\alpha\gamma$ and $\alpha\beta$, respectively, gives

$$\frac{\beta_1\gamma_2 - \beta_2\gamma_1}{\alpha} = \frac{\gamma_1\alpha_2 - \gamma_2\alpha_1}{\beta} = \frac{\alpha_1\beta_2 - \alpha_2\beta_1}{\gamma} \equiv \mu;$$

$$\therefore \qquad \begin{array}{l}\mu\alpha = \beta_1\gamma_2 - \beta_2\gamma_1;\\ \mu\beta = \gamma_1\alpha_2 - \gamma_2\alpha_1;\\ \mu\gamma = \alpha_1\beta_2 - \alpha_2\beta_1.\end{array}$$

Taking the sum of the squares of these equations,

$$\mu^2 = (\beta_1\gamma_2 - \beta_2\gamma_1)^2 + (\gamma_1\alpha_2 - \gamma_2\alpha_1)^2 + (\alpha_1\beta_2 - \alpha_2\beta_1)^2, \quad (7)$$

which is the square of the sine of the angle between the given lines (§ 218).

The direction-cosines α, β and γ are therefore

$$\left.\begin{array}{l}\alpha = \dfrac{\beta_1\gamma_2 - \beta_2\gamma_1}{\sin v};\\[2ex] \beta = \dfrac{\gamma_1\alpha_2 - \gamma_2\alpha_1}{\sin v};\\[2ex] \gamma = \dfrac{\alpha_1\beta_2 - \alpha_2\beta_1}{\sin v};\end{array}\right\} \qquad (8)$$

v being the angle between the given lines. Thus the direction of the required line is completely determined.

To complete the solution, we must find the co-ordinates of some point of the line. Let us then put

(a, b, c) the point in which the required line intersects the first of the given lines. The equations of the required line may then be written

$$\left.\begin{aligned} x &= a + \alpha\rho; \\ y &= b + \beta\rho; \\ z &= c + \gamma\rho. \end{aligned}\right\} \qquad (b)$$

Let us also put

ρ_1, the distance of the point (a, b, c) from (x_1, y_1, z_1) on the first given line;

ρ_2, the distance from (x_2, y_2, z_2) on the second given line to the point in which the required line intersects it;

ρ_0, the distance of the points of intersection, that is, the length of the shortest line between the given lines.

Then, equating the expressions for the co-ordinates of the points of intersection on the two lines, we have the six equations

$$\left.\begin{array}{ll} \left.\begin{aligned} a &= x_1 + \alpha_1\rho_1; \\ b &= y_1 + \beta_1\rho_1; \\ c &= z_1 + \gamma_1\rho_1. \end{aligned}\right\} & \text{Intersection of required line with first given line.} \\ \left.\begin{aligned} a + \alpha\rho_0 &= x_2 + \alpha_2\rho_2; \\ b + \beta\rho_0 &= y_2 + \beta_2\rho_2; \\ c + \gamma\rho_0 &= z_2 + \gamma_2\rho_2. \end{aligned}\right\} & \text{Intersection of required line with second given line.} \end{array}\right\} \qquad (c)$$

These six equations suffice to completely determine the six unknown quantities, a, b, c, ρ_0, ρ_1, ρ_2. First subtracting corresponding equations in the two sets, we eliminate a, b and c, and have three equations which we may write in the form

$$\left.\begin{aligned} \alpha\rho_0 + \alpha_1\rho_1 - \alpha_2\rho_2 &= x_2 - x_1, \\ \beta\rho_0 + \beta_1\rho_1 - \beta_2\rho_2 &= y_2 - y_1, \\ \gamma\rho_0 + \gamma_1\rho_1 - \gamma_2\rho_2 &= z_2 - z_1, \end{aligned}\right\} \qquad (d)$$

and which contain only the three unknown quantities ρ_0, ρ_1 and ρ_2. Multiplying the equations in order by α, β and γ, taking their sum and referring to the relations (6), we have

$$\rho_0 = \alpha(x_2 - x_1) + \beta(y_2 - y_1) + \gamma(z_2 - z_1). \qquad (9)$$

From the manner in which ρ_0 has been defined, it is equal to the shortest distance between the two lines (1) and (2), because it is the distance from the point (a, b, c) to the point in which the shortest line intersects the second line.

If we substitute for α, β and γ their values (8), we have

$$\rho_0 = \frac{(\beta_1\gamma_2-\beta_2\gamma_1)(x_2-x_1)+(\gamma_1\alpha_2-\gamma_2\alpha_1)(y_2-y_1)+(\alpha_1\beta_2-\alpha_2\beta_1)(z_2-z_1)}{\sin v}. \quad (10)$$

Again, multiplying the equations (d) by α_1, β_1 and γ_1, and adding, and then by α_2, β_2 and γ_2, and adding, we find

$$\rho_1 - \rho_2 \cos v = \alpha_1(x_2 - x_1) + \beta_1(y_2 - y_1) + \gamma_1(z_2 - z_1) \equiv r_1;$$
$$\rho_1 \cos v - \rho_2 = \alpha_2(x_2 - x_1) + \beta_2(y_2 - y_1) + \gamma_2(z_2 - z_1) \equiv r_2.$$

Hence

$$\rho_1 = \frac{r_1 - r_2 \cos v}{\sin^2 v};$$

$$\rho_2 = \frac{r_1 \cos v - r_2}{\sin^2 v}.$$

To find a, b and c, we have only to substitute the value of ρ_1 in (c), which gives

$$\left.\begin{aligned} a &= x_1 + \frac{\alpha_1 r_1 - \alpha_1 r_2 \cos v}{\sin^2 v}; \\ b &= y_1 + \frac{\beta_1 r_1 - \beta_1 r_2 \cos v}{\sin^2 v}; \\ c &= z_1 + \frac{\gamma_1 r_1 - \gamma_1 r_2 \cos v}{\sin^2 v}. \end{aligned}\right\} \quad (11)$$

The values of a, b, c, α, β and γ in (11) and (8) being substituted in (b), the equations of the shortest line are complete.

249. *Condition of Intersection.* Since ρ_0 in (9) expresses the shortest distance between the two lines, the condition that the lines shall intersect is found by putting $\rho = 0$. Substituting for α, β and γ their values (8), this condition gives

$$(\beta_1\gamma_2 - \beta_2\gamma_1)(x_2 - x_1) + (\gamma_1\alpha_2 - \gamma_2\alpha_1)(y_2 - y_1) + (\alpha_1\beta_2 - \alpha_2\beta_1)(z_2 - z_1) = 0.$$

If, instead of α, β and γ, we use the quantities l, m and n, we must, from the proportionality of these factors to α, β and γ, have α, β and γ equal respectively to l, m and n, each multiplied by the same factor.

If we call p and q these factors in the cases of α_1, β_1, γ_1 and α_2, β_2, γ_2 respectively, we have

$$\begin{aligned} \beta_1\gamma_2 - \beta_2\gamma_1 &= pq(m_1n_2 - m_2n_1), \\ \gamma_1\alpha_2 - \gamma_2\alpha_1 &= pq(n_1l_2 - l_2n_1), \\ \alpha_1\beta_2 - \alpha_2\beta_1 &= pq(l_1m_2 - l_2m_1), \end{aligned}$$

and the condition of intersection becomes

$$(m_1n_2-m_2n_1)(x_2-x_1)+(n_1l_2-n_2l_1)(y_2-y_1)+(l_1m_2-l_2m_1)(z_2-z_1)=0. \quad (12)$$

250. PROBLEM. *To find the point in which a line intersects a surface.*

Since the point of intersection lies on the line, there will be a definite value of ρ corresponding to it. This value of ρ, being substituted in the equation of the line, will give values of the co-ordinates x, y and z which, if ρ is properly taken, will satisfy the equation of the surface. We therefore proceed as follows:

Calling, for the moment, (a, b, c) any one point of the given line, we substitute in the equation of the surface, for x, y and z, the expressions

$$x = a + l\rho; \qquad y = b + m\rho; \qquad z = c + n\rho. \qquad (a)$$

The equation of the surface will then contain no unknown quantity except ρ, and is to be solved so as to get an expression for ρ which shall satisfy it.

This expression being substituted in the equations (a) will give the required values of the co-ordinates of the point of intersection.

If the equation in ρ is of a higher degree than the first, there will be several values of ρ, and therefore several points of intersection.

EXAMPLE. Find the point in which the line

$$\begin{aligned} x &= 2 + 2\rho, \\ y &= 3 - 2\rho, \\ z &= 5 - \rho, \end{aligned}$$

intersects the plane

$$2x - 3y - z + 8 = 0.$$

Substituting the values of the co-ordinates in the equation of the plane, we find

$$-10 + 11\rho + 8 = 0,$$

which gives $$\rho = \frac{2}{11};$$

whence

$$x = 2\tfrac{4}{11}, \qquad y = 2\tfrac{7}{11}, \qquad z = 4\tfrac{9}{11},$$

are the co-ordinates of the point of intersection.

The same general method applies whenever points fulfilling any condition whatever are to be found on one or more lines. Each line must have its own value of ρ, which we may distinguish from the values for other lines by accents or subscript numbers. The values of the co-ordinates, expressed in terms of ρ, are to be substituted in each condition, and equations with the ρ's as the only unknown quantities will thus be formed.

EXERCISES.

1. Write the equations of the sides of the triangle whose vertices are at the points (1, 2, 3), (3, 2, 1) and (2, 3, 1), and find the angles of the triangle.

Ans., in part. 30°, 60°, 90°.

2. Find the points in which the line joining the points (1, 2, 3) and (2, 3, 4) intersects the co-ordinate planes.

Ans. (0, 1, 2); (− 1, 0, 1); (− 2, − 1, 0).

3. Write the symmetrical equations of the line passing through the point (a, b, c) and perpendicular to the plane

$$px + qy + rz = 0.$$

4. An equilateral triangle has one vertex in each co-ordinate plane, at the distance h from each of the axes lying in that plane. Write the equations of each of its sides, taking the middle point of each side as the point from which ρ is measured.

Ans., in part. $$\left.\begin{aligned} x &= h; \\ y &= \tfrac{1}{2}h + \rho; \\ z &= \tfrac{1}{2}h - \rho. \end{aligned}\right\} \text{Equations of one of the sides.}$$

5. In what points does the line of intersection of the two planes

$$x + y - z = 7,$$
$$x - y + 2z = 1,$$

intersect the co-ordinate planes?

Ans. $(4, 3, 0)$; $(0, 15, 8)$; $(5, 0, -2)$.

6. Express the point in which the line (4), § 246, intersects the plane $Lx + My + Nz = 0$.

$$\textit{Ans., in part.}\ x = \frac{M(mx_0 - ly_0) + N(nx_0 - lz_0)}{lL + Mm + Nn}.$$

7. Write the symmetrical equations of the line of intersection of the two planes

$$x + 2y - 3z - 5 = 0,$$
$$2x - y + 2z + 7 = 0,$$

taking as the zero point of the line that in which it intersects the plane XY.

$$\textit{Ans.}\ x = -\frac{9}{5} - \rho; \quad y = \frac{17}{5} + 8\rho; \quad z = 0 + 5\rho.$$

In cases where direction-vectors appear as unknown quantities in equations, there will be but two equations for the three vectors. In this case we determine any two in terms of the third, and assign to the latter such value as will give the simplest form to the results.

8. Write the equations of the line passing through the point (3, 1, 5) and intersecting the axis of X perpendicularly.

9. Write the equations of the line passing through the point (a, b, c) and parallel to each of the planes

$$x + y - 2z = 0;$$
$$x - y + z = 0.$$

$$\textit{Ans.}\ x = a + \rho;$$
$$y = b + 3\rho;$$
$$z = c + 2\rho.$$

10. Find the condition that the line (4), § 246, shall intersect the axis of Z. *Ans.* $mx_0 = ly_0$.

The condition requires that the points in which the line intersects the planes of XZ and YZ respectively shall be the same, that is, correspond to the same value of ρ.

11. Find the equation of the line passing through the point (5, 2, 4) and intersecting perpendicularly that line through the origin whose direction-vectors are $l = 1$, $m = 2$, $n = 2$. *Ans.* $x = 5 - 14\rho$; $y = 2 + 8\rho$; $z = 4 - \rho$.

Write the symmetrical equations of the given and the required line, calling ρ' the variable for the one line, and ρ for the other. The condition that some one point (x, y, z) shall satisfy the equations of *both* lines then gives three equations of condition between l, m, n, ρ' and ρ, and the condition of perpendicularity gives a fourth.

12. Deduce the condition that two lines shall intersect by the principle that there must then be *one* set of values of x, y and z which shall satisfy the equations of *both* lines, these values of the co-ordinates being given in terms of one value of ρ for the one line, and another value for the other line.

The condition gives three equations between the two quantities ρ (on one line) and ρ' (on the other), and the values of ρ and ρ' must be the same, whether we derive them from one or another pair of the equations.

13. Write the equation of the plane which contains the two intersecting lines

$$\begin{aligned} x &= a + \rho; & x &= a - \rho; \\ y &= b - \rho; & y &= b - 2\rho; \\ z &= c - 2\rho; & z &= c - 3\rho. \end{aligned}$$

Ans. $x - 5y + 3z - a + 5b - 3c = 0$.

Note that the condition that a plane shall contain or be parallel to a line is the same as that a line shall be parallel to or lie in a plane.

14. Find that plane which is parallel to each of the lines

$$\begin{aligned} x &= a - 2\rho; & x &= a' + \rho; \\ y &= b - \rho; & y &= b' + 2\rho; \\ z &= c + \rho; & z &= c' - \rho; \end{aligned}$$

and equidistant from them. Also find the common distance.

Ans. $2x + 2y + 6z - a - a' - b - b' - 3c - 3c' = 0$.

$$\text{Common dist.} = \frac{a - a' + b - b' + 3(c - c')}{2\sqrt{11}}.$$

CHAPTER IV.

QUADRIC SURFACES.

General Properties of Quadrics.

251. *Def.* A **quadric surface** is the locus of a point in space whose co-ordinates are required to satisfy an equation of the second degree.

REMARK. A quadric surface is called a *quadric* simply.

The most general form of an equation of the second degree between the co-ordinates is

$$gx^2 + hy^2 + kz^2 + 2g'yz + 2h'zx + 2k'xy \\ + 2g''x + 2h''y + 2k''z + d = 0, \quad (1)$$

because the terms in this equation include all powers and products of the co-ordinates x, y and z up to those of the second degree.

The number of coefficients, as written, is ten. But since, by division, we may reduce any coefficient to unity, their number is, in effect, equivalent to nine. Hence:

THEOREM I. *Nine conditions are necessary to determine a quadric in space.*

REMARK. In discussing the equation (1), we regard the coefficients g, h, k, g', etc., as given constants, unless otherwise expressed.

We may trace out certain analogies between the quadric and conic, by treating the former in the same general manner in which we treated the latter in Part I.

252. *Intersections of a Quadric with a Straight Line.* Let the equations of the line be

$$\left.\begin{aligned} x &= x_0 + l\rho; \\ y &= y_0 + m\rho; \\ z &= z_0 + n\rho. \end{aligned}\right\} \qquad (a)$$

The problem now is, What values of x, y and z satisfy both these equations and (1)? (Cf. § 250.) If we substitute these expressions for x, y and z in (1), thus:

$$\begin{gathered} gx^2 = g(x_0^2 + 2lx_0\rho + l^2\rho^2), \\ \text{etc.} \qquad\qquad \text{etc.}, \end{gathered}$$

we shall have an equation in which all the quantities except ρ will be supposed given. Hence ρ can be determined from the equation. When this is done, the values of the co-ordinates of the point or points of intersection are found by substituting the values of ρ in (a). Now the equation in ρ will be of the second degree, and will therefore have two roots, which may be real, imaginary or equal. Hence:

THEOREM II. *Every straight line intersects a quadric in two real, imaginary or coincident points.*

253. *Centre of Quadric.* Let us next change the origin to a point whose co-ordinates have the arbitrary values A, B, C. If we distinguish the co-ordinates referred to the new system by accents, we shall have (§ 219)

$$\begin{aligned} x &= x' + A; \\ y &= y' + B; \\ z &= z' + C. \end{aligned}$$

Substituting these values in the general equation (1), it becomes

$$\begin{aligned} gx'^2 &+ hy'^2 + kz'^2 + 2g'y'z' + 2h'z'x' + 2k'x'y' \\ &+ 2(gA + k'B + h'C + g'')x' \\ &+ 2(hB + g'C + k'A + h'')y' \\ &+ 2(kC + h'A + g'B + k'')z' \\ &+ gA^2 + hB^2 + kC^2 + 2g'BC + 2h'CA + 2k'AB \\ &\qquad + 2g''A + 2h''B + 2k''C + d = 0. \qquad (1') \end{aligned}$$

Let us now determine the co-ordinates A, B, C of the new origin by the condition that the coefficients of x', y' and z' shall all vanish. To do this we have to solve the equations

$$\left.\begin{array}{l} gA + k'B + h'C = -g''; \\ k'A + hB + g'C = -h''; \\ h'A + g'B + kC = -k''; \end{array}\right\} \qquad (2)$$

regarding A, B and C as the unknown quantities. Since there are as many equations as unknown quantities, the solution will, *in general*, be possible.

Let us now suppose the equations (2) to be solved, and the resulting values of A, B and C to be substituted in (1′). Let us also put

d', the absolute terms in (1′).

Then, omitting accents, the equation (1′) of the quadric reduces to

$$gx^2 + hy^2 + kz^2 + 2g'yz + 2h'zx + 2k'xy + d' = 0. \qquad (3)$$

From this equation we may deduce a second fundamental property of the quadric. If (x, y, z) be any values of the co-ordinates which satisfy (3), it is evident that $(-x, -y, -z)$ will also satisfy it. Hence, if one of these points is on the quadric, the other will also be on it. But the line joining these points passes through the origin, and is bisected by the origin, that is, by the point whose co-ordinates, referred to the original system, are (A, B, C). Since (x, y, z) may be any point on the quadric, we conclude:

Theorem III. *For every quadric there is, in general, a point which bisects every chord passing through it.*

Def. That point which bisects every chord passing through it is called the **centre** of the quadric.

A chord through the centre is called a **diameter** of the quadric.

254. *Section of a Quadric by a Plane.* To investigate the equation of the plane curve in which any plane intersects the quadric, we may take a pair of co-ordinate axes in the cutting plane. This we do by simply transforming the equation to one referred to new axes; and we may, in the first

place, leave the origin unchanged. Accenting the new co-ordinates, the equations of transformation will be (§ 220)

$$\begin{aligned} x &= \alpha x' + \beta y' + \gamma z'; \\ y &= \alpha' x' + \beta' y' + \gamma' z'; \\ z &= \alpha'' x' + \beta'' y' + \gamma'' z'; \end{aligned}$$

α, β, γ, etc., being the direction-cosines of the new co-ordinate axes relatively to the old ones.

Now when we substitute these expressions in the general equation (1) and arrange the terms according to the powers and products of x', y' and z', we shall have a new equation of the same form as (1), that is, one containing terms in x'^2, y'^2, z'^2, $x'y'$, etc.; the only change being that the coefficients g, h, k, g', etc., have new values. We may therefore, without any loss of generality, take the equation (1) as representing the transformed equation, and consider the section of the surface which it represents by a plane parallel to any one of the co-ordinate planes, XY for example. Let us then suppose $z = c$ in (1). The equation of the section of the quadric by the plane $z = c$ will be, omitting the accents,

$$gx^2 + hy^2 + 2k'xy + 2(h'c + g'')x + 2(g'c + h'')y \\ + kc^2 + 2k''c + d = 0.$$

This is the equation of a conic section. Hence:

THEOREM IV. *Every plane section of a quadric is a conic section.*

It is also shown in § 198 that all conics whose equations have the same coefficients in x^2, xy and y^2 are similar and similarly placed. Now, in the above equation, the coefficients g, h and $2k$ remain unaltered, however c may change; that is, however we may change the position of the cutting plane, so long as it remains parallel to the plane of XY. Hence:

THEOREM. V. *All sections of a quadric by parallel planes are similar conics and have their principal axes parallel.*

Cor. If any plane section of a quadric is a circle, all sections parallel to it are circles.

EXERCISES.

1. Find the centre of the quadric

$$x^2 + hy^2 + z^2 + nyz + mx = 0.$$

2. Write the equation of the locus of the point required to be equally distant from the origin and from the plane

$$\alpha x + \beta y + \gamma z - p = 0. \qquad (\alpha^2 + \beta^2 + \gamma^2 = 1.)$$

3. Write the equation of the locus of the point equally distant from the origin and from the plane

$$cx + c'y - p = 0, \qquad (c^2 + c'^2 = 1.)$$

and show that its centre is at infinity.

255. *Conjugate Axes and Planes.* Consider this problem:

To find the locus of the middle points of all chords of a quadric parallel to any fixed line, and therefore to each other.

Let the equation of any one of the chords be

$$x = x_0 + l\rho;$$
$$y = y_0 + m\rho;$$
$$z = z_0 + n\rho.$$

If l, m and n remain constant, then, by asigning all values to x_0, y_0 and z_0, these equations may represent any system of lines parallel to each other. Now, we find the two points in which any one of these lines intersects the surface by the process of § 250; namely, we put in the equation (3) of the surface

$$x^2 = x_0^2 + 2lx_0\rho + l^2\rho^2;$$
$$y^2 = y_0^2 + 2my_0\rho + m^2\rho^2;$$
$$z^2 = z_0^2 + 2nz_0\rho + n^2\rho^2;$$
$$yz = y_0z_0 + (ny_0 + mz_0)\rho + mn\rho^2;$$
$$zx = z_0x_0 + (nx_0 + lz_0)\rho + nl\rho^2;$$
$$xy = x_0y_0 + (ly_0 + mx_0)\rho + lm\rho^2.$$

For brevity, let us represent the result of substituting these values in (3) in the form

$$A + B\rho + C\rho^2 = 0. \qquad (a)$$

Now, we may choose for (x_0, y_0, z_0) any point on the chord. Let us choose the middle point. This point will be determined by the condition that the two values of ρ from the quadratic equation (a) shall be equal, with opposite algebraic signs. The condition for this result is $B = 0$. That is, writing for B its value, the condition will be

$$glx_0 + hmy_0 + knz_0 + g'(ny_0 + mz_0) + h'(nx_0 + lz_0) \\ + k'(ly_0 + mx_0) = 0.$$

This, then, is the equation which the middle point (x_0, y_0, z_0) must satisfy as x_0, y_0 and z_0 vary. Being of the first degree, it is the equation of a plane, and, having no absolute term, the plane passes through the origin, that is, the centre of the quadric. Hence:

THEOREM. VI. *The locus of the middle points of a system of parallel chords of a quadric is a plane through the centre.*

Def. A plane through the centre of a quadric is called a **diametral plane.**

That diametral plane which bisects all chords parallel to a diameter is said to be **conjugate** to such diameter, and the diameter is **conjugate** to the plane.

That diameter whose direction-vectors are l, m, n, that is, whose equations are

$$x = l\rho,$$
$$y = m\rho,$$
$$z = n\rho,$$

will be called the diameter (l, m, n).

REMARK. If we call any diameter A, we may call the conjugate diametral plane A'.

The above equation of the diametral plane may be written out thus, the subscript zeros being omitted:

$$(gl + k'm + h'n)x + (k'l + hm + g'n)y \\ + (h'l + g'm + kn)z = 0. \qquad (4)$$

That is, *this plane is conjugate to the diameter* (l, m, n).

THEOREM. VII. *If a diameter B lie in a plane A', the conjugate diametral plane B' will contain the diameter A, conjugate to A'.*

Proof. Let the equation (4) represent the diametral plane A', and let the diameter B be (λ, μ, ν). By § 247, the condition that this diameter shall lie in the plane (4) is

$$(gl + k'm + h'n)\lambda + (k'l + hm + g'n)\mu + (h'l + g'm + kn)\nu = 0,$$

or, rearranging the terms,

$$(g\lambda + k'\mu + h'\nu)l + (k'\lambda + h\mu + g'\nu)m + (h'\lambda + g'\mu + k\nu)n = 0.$$

But (§ 247) this is the condition that the diameter (l, m, n), or A, shall lie in the plane

$$(g\lambda + k'\mu + h'\nu)x + (k'\lambda + h\mu + g'\nu)y + (h'\lambda + g'\mu + k\nu)z = 0,$$

which, by comparison with (4), is seen to represent the plane conjugate to the diameter (λ, μ, ν), or B; that is, the plane B'. Hence this plane contains the diameter A. Q. E. D.

Scholium. Having two conjugate diameters, A and B, with their diametral planes, A' and B', arranged as in this theorem, the intersection of the planes A' and B' will determine a third diameter, which we may call C. Then, because C lies in both the planes A' and B', its conjugate plane C' will, by Theorem VII., pass through both A and B. Thus we shall have a system of three diametral planes whose intersections will form three diameters, and each plane will bisect all chords parallel to its conjugate diameter.

These three lines and planes are called a **system of conjugate axes and diametral planes.**

256. *Change in the Direction of the Axes.* To simplify the equation (3) still further, let us change the direction of the axes of co-ordinates, leaving the origin at the centre. This we do by the substitution

$$\begin{aligned} x &= \alpha x' + \beta y' + \gamma z'; \\ y &= \alpha' x' + \beta' y' + \gamma' z'; \\ z &= \alpha'' x' + \beta'' y' + \gamma'' z'. \end{aligned}$$

If we substitute these values in (3), we shall have an equation

the terms of which we can arrange according to the powers and products of x', y' and z'; namely,

$$x'^2, \quad y'^2, \quad z'^2, \quad y'z', \quad z'x', \quad x'y'.$$

We then suppose the values of the direction-cosines α, β, γ, α', etc., to be so determined that the coefficients of $y'z'$, $z'x'$ and $x'y'$ shall all three vanish. This will require three equations of condition to be satisfied, which, with the six relations (14) of §220, will completely determine the nine direction-cosines.*

These cosines being determined, the coefficients of x'^2, y'^2 and z'^2 will all become known quantities, while the products $y'z'$, etc., will disappear. Thus, omitting once more the accents from the co-ordinates, the equation (3) will be reduced to the form

$$l'x^2 + m'y^2 + n'z^2 + d' = 0,$$

or

$$-\frac{l'}{d'}x^2 - \frac{m'}{d'}y^2 - \frac{n'}{d'}z^2 = 1, \qquad (5)$$

l', m' and n' being known quantities, functions of the original coefficients in (1). It will be seen that the absolute term d' remains unaltered by this transformation.

The several quantities $\frac{l'}{d'}$, $\frac{m'}{d'}$, etc., may be either positive or negative, according to the values of the coefficients which enter into the original equation (1).

257. *Principal Axes.* Let us put A for the value of $\frac{d'}{l'}$ taken *positively;* the first term of (5) will be $\pm \frac{x^2}{A}$, according to whether it is positive or negative. If then we put

$$a = \sqrt{A} = \sqrt{\pm \frac{d'}{l'}},$$

the term will become $\pm \frac{x^2}{a^2}$

---

* The equations obtained in this way are too complex for convenient management, and the actual values of the direction-cosines must be found by the differential calculus, or by an application of the algebra of linear substitutions. We must therefore, at present, be contented with showing the possibility of the solution, which is all that is necessary for our immediate purpose.

In the same way the other terms can be reduced to the form $\pm \frac{y^2}{b^2}$ and $\pm \frac{z^2}{c^2}$. Thus the general equation of the quadric can finally be reduced to the form

$$\pm \frac{x^2}{a^2} \pm \frac{y^2}{b^2} \pm \frac{z^2}{c^2} = 1. \tag{6}$$

Def. The quantities a, b and c in this equation are called the **principal axes** of the quadric.

The Three Classes of Quadrics.

258. There are now four possible cases, omitting the exceptional ones in which a, b, or c is zero or infinity.

CASE I. The coefficients of the first member of (6) all positive.

CASE II. Two coefficients positive and one negative.

CASE III. One coefficient positive and two negative.

CASE IV. All the coefficients negative.

In the last case no real values of the co-ordinates can satisfy the equation, because the terms, being themselves squares, are essentially positive, and therefore with the minus signs essentially negative. Hence there can be no real surface to represent the equation. But in the other three cases there will be real loci. Hence

There are three general classes of real quadrics.

259. CLASS I. *The Ellipsoid.* In Case I. the equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \tag{7}$$

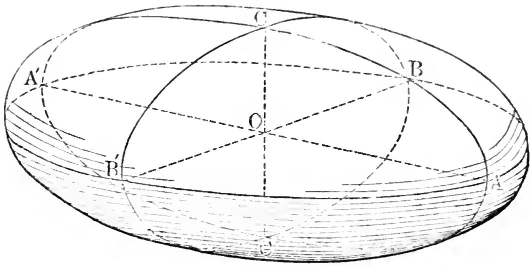
Let us first investigate the limiting values of the co-ordinates. Writing the equation in the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{z^2}{c^2},$$

we see that when $-c > z > +c$, the co-ordinates x and y cannot both be real. Hence the surface is wholly included between the two planes $z = +c$ and $z = -c$.

In the same way it is shown that the surface is included between the planes $x = +a$ and $x = -a$, and also between the planes $y = +b$ and $y = -b$. Hence it is bounded in every direction.

Because its sections by a plane are of the second order, and limited in extent, they must all be ellipses. Hence the surface is called an ellipsoid.



If we suppose $z = \pm c$, we have $x = 0$ and $y = 0$, as the only values of x and y which can satisfy the equation. Hence each of the two planes $z = +c$ and $z = -c$ meets the surface in a single point on the axis of Z, and is therefore tangent to the surface. Extending the same proof to the other two co-ordinates, we reach the conclusion:

The ~~*eight*~~ six *planes* $x = +a$, $x = -a$, $y = +b$, $y = -b$, $z = +c$ *and* $z = -c$ *are all tangents to the ellipsoid at the points which lie on the axes at the distances* $\pm a$, $\pm b$ *and* $\pm c$ *from the origin.*

These six planes form the faces of a rectangular parallelopiped whose edges are respectively $2a$, $2b$ and $2c$. Each pair of parallel faces being at equal distances on the two sides of the origin, and parallel to the corresponding axes, these axes intersect the faces in their centres. Hence:

THEOREM VIII. *Every ellipsoid may be inscribed in a rectangular parallelopiped whose surface it will touch in the centre of each face.*

260. CLASS II. *The Hyperboloid of One Nappe.* Let us take that form of the equation (7) in which one of the three

terms of the first member is negative. Suppose this term to be that in z. The equation is then

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1, \tag{8}$$

which we may write in the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 + \frac{z^2}{c^2}. \tag{8'}$$

Let us now find the curve in which the surface intersects the plane of XY. This we do by putting $z = 0$, which gives at once the equation of an ellipse whose major axes are a and b. Hence:

THEOREM IX. *The hyperboloid of one nappe intersects the plane of XY in an ellipse whose axes are the same as the axes a and b of the surface.*

This ellipse is called the **ellipse of the gorge.**

Let us next find the curve in which the surface intersects a plane parallel to that of XY and at a distance k from it. The equation of such a plane is

$$z = k.$$

Substituting this constant value of z, and putting, for brevity,

$$h^2 \equiv 1 + \frac{k^2}{c^2}, \tag{a}$$

the equation (8′) reduces to

$$\frac{x^2}{h^2a^2} + \frac{y^2}{h^2b^2} = 1.$$

This is the equation of an ellipse whose axes are ha and hb. Whatever the value of h, the ratio of these axes will be $a : b$, so that the ellipses will be similar. Hence:

THEOREM X. *The hyperboloid of one nappe cuts all planes perpendicular to its axis of Z in similar ellipses.*

The equation (a) shows that h exceeds unity and increases with positively or negatively increasing values of k. Hence

The ellipses in which the hyperboloid of one nappe cuts planes perpendicular to its axis of Z are larger the farther the planes are from the centre.

To find the curves in which the surface intersects planes parallel to the other co-ordinate planes, we transpose either the term in x or that in y, thus putting the equation in the form

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 - \frac{y^2}{b^2}. \tag{9}$$

Assigning any constant value to y, we see that the equation is that of an hyperbola, and we may show, as in the case of the other section, that these hyperbolas are all similar. But there is one remarkable case, namely, that in which the equation of the intersecting plane is

$$y = \pm b.$$

The equation of the intersection then becomes

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 0,$$

or $$(cx - az)(cx + az) = 0,$$

which is the equation of a pair of straight lines.

This result will be generalized hereafter.

261. Class III. *Hyperboloid of Two Nappes.* Let two of the terms in (6) be negative. By taking the terms in x and y as negative, and then changing the sign of each member of the equation, it may be written in the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2} - 1. \tag{10}$$

If $c > z > -c$, the second member will be negative and the equation can be satisfied by no real values of x and y. When z is on either side of the limits $\pm c$, there will be real values of x and y. Hence *the surface is composed of two distinct sheets, or nappes, separated at their nearest points by the space 2c.* This surface is therefore called the **hyperboloid of two nappes.**

We readily see that the plane $z = k$, parallel to the plane of XY, intersects the surface in a real ellipse whenever $k > c$.

We also show, as in the last section, that the planes $x = k$ and $y = k$ intersect it in hyperbolas.

Tangent and Polar Lines and Planes to a Quadric.

262. Since the equation of the general quadric surface may be reduced to one of the three forms just considered, we may, without loss of generality, consider the equations (6) as representing every such surface. Moreover, we may, in beginning, restrict ourselves to the first form,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \tag{11}$$

because the fact that any of the three terms of the first member has the negative sign may be indicated by substituting

$$-a^2, \ -b^2 \text{ or } -c^2 \qquad \text{for} \qquad a^2, b^2 \text{ or } c^2.$$

Def. A **tangent line** to a surface is a line which passes through two coincident points of the surface.

Problem. *To find the condition that a line shall touch a surface of the second order at the point* (x_1, y_1, z_1) *on that surface.*

Solution. Since the line passes through the point (x_1, y_1, z_1) of tangency, its equations may be written in the form

$$\left.\begin{aligned} x &= x_1 + l\rho; \\ y &= y_1 + m\rho; \\ z &= z_1 + n\rho. \end{aligned}\right\} \tag{a}$$

So long as l, m and n are unrestricted, these equations may represent any line through the point (x_1, y_1, z_1).

To find the points in which the line meets the surface (11), we must substitute these values of x, y and z in the equation of the surface. Doing this, and arranging the equations in powers of ρ, we have the condition

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} - 1 + 2\rho\left(\frac{lx_1}{a^2} + \frac{my_1}{b^2} + \frac{nz_1}{c^2}\right)$$
$$+ \rho^2\left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right) = 0. \tag{b}$$

Since the point (x_1, y_1, z_1) lies on the surface by hypothesis, we have

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} - 1 = 0, \qquad (c)$$

from which it follows that $\rho = 0$ is a root of (b). This gives the point (x_1, y_1, z_1) as one of the points, as it ought to. Dividing by ρ, the equation becomes

$$\left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right)\rho + 2\left(\frac{lx_1}{a^2} + \frac{my_1}{b^2} + \frac{nz_1}{c^2}\right) = 0,$$

which gives, for the other value of ρ,

$$\rho = -2\left(\frac{lx_1}{a^2} + \frac{my_1}{b^2} + \frac{nz_1}{c^2}\right) \div \frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}. \qquad (d)$$

We have hitherto subjected the line (a) to no restriction except that of passing through the point (x_1, y_1, z_1). The equation (d) gives the value of ρ in terms of l, m and n for the second point in which the line intersects the surface.

Now, the problem requires that this second point shall coincide with the first one, that is, that $\rho = 0$ in (d). This gives

$$\frac{lx_1}{a^2} + \frac{my_1}{b^2} + \frac{nz_1}{c^2} = 0 \qquad (12)$$

as the required condition that the line A shall touch the quadric at the point (x_1, y_1, z_1).

All the quantities except l, m and n in this equation being regarded as given constants, it constitutes a linear equation between l, m and n. Hence, by § 247 (b), it requires that the tangent line lie in the plane

$$\frac{x_1}{a^2}(x - x_1) + \frac{y_1}{b^2}(y - y_1) + \frac{z_1}{c^2}(z - z_1) = 0,$$

which, by (c), readily reduces to

$$\frac{x_1x}{a^2} + \frac{y_1y}{b^2} + \frac{z_1z}{c^2} - 1 = 0. \qquad (13)$$

Hence we reach the conclusion:

THEOREM XI. *All straight lines touching a quadric surface at the same point lie in a certain plane passing through that point.*

Def. The plane containing all lines tangent to a surface at the same point is called a **tangent plane** to the surface, and is said to **touch** the surface at that point.

263. *Lines upon the Hyperboloid of One Nappe.* The result of §260, that a plane may intersect an hyperboloid in a pair of straight lines, is a special case of the following theorem:

THEOREM XII. *Through every point upon the hyperboloid of one nappe pass two straight lines which lie wholly on the surface, and which form the intersection of the plane tangent at that point with the surface.*

Proof. We may write the equation (9) of the hyperboloid in the form

$$\left(\frac{x}{a}+\frac{z}{c}\right)\left(\frac{x}{a}-\frac{z}{c}\right)=\left(1+\frac{y}{b}\right)\left(1-\frac{y}{b}\right). \qquad (b)$$

Now, putting λ for an arbitrary constant, let us consider the two planes whose equations are:

$$\left.\begin{aligned}&\text{First plane,} \quad \frac{x}{a}+\frac{z}{c}=\lambda\left(1+\frac{y}{b}\right);\\ &\text{Second plane,} \quad \frac{x}{a}-\frac{z}{c}=\frac{1}{\lambda}\left(1-\frac{y}{b}\right).\end{aligned}\right\} \qquad (c)$$

Every set of values of x, y and z which satisfies these two equations simultaneously satisfies the equation (b) of the surface, as we readily find by multiplication. But all such values belong to the line in which the two planes intersect. Hence this line lies wholly in the surface.

We have next to show that, by giving a suitable value to λ, this line may pass through any point of the surface. Let us put (x_1, y_1, z_1), the point through which the line is to pass.

The factor λ must then satisfy the two equations

$$\frac{x_1}{a}+\frac{z_1}{c}=\lambda\left(1+\frac{y_1}{b}\right);$$
$$\frac{x_1}{a}-\frac{z_1}{c}=\frac{1}{\lambda}\left(1-\frac{y_1}{b}\right);$$

whence

$$\lambda = \frac{x_1}{a} + \frac{z_1}{c} \div 1 + \frac{y_1}{b} = 1 - \frac{y_1}{b} \div \frac{x_1}{a} - \frac{z_1}{c}. \qquad (d)$$

These two equations give the same value of λ when x_1, y_1 and z_1 are required to satisfy the equation of the surface. Substituting the second value of λ in the first equation (c) of the line of intersection, and the first value in the second, these equations readily reduce to

$$\left(\frac{x_1}{a^2} - \frac{z_1}{ac}\right)x + \left(\frac{y_1}{b^2} - \frac{1}{b}\right)y - \left(\frac{z_1}{c^2} - \frac{x_1}{ac}\right)z = 1 - \frac{y_1}{b}$$

$$\left(\frac{x_1}{a^2} + \frac{z_1}{ac}\right)x + \left(\frac{y_1}{b^2} + \frac{1}{b}\right)y - \left(\frac{z_1}{c^2} + \frac{x_1}{ac}\right)z = 1 + \frac{y}{b}.$$

Taking half the sum and half the difference of these equations, they become

$$\left.\begin{aligned} \frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} - \frac{z_1 z}{c^2} &= 1; \\ \frac{z_1 x}{ac} + \frac{y}{b} \quad - \frac{x_1 z}{ac} &= \frac{y_1}{b}; \end{aligned}\right\} \qquad (e)$$

which are still the equations of the line in question, in another form. But the first of these equations is that of the tangent plane. Hence the line lies in the tangent plane as well as on the surface, and therefore forms the intersection of the plane with the surface.

The other line through (x_1, y_1, z_1) is found, in the same way, to be given by the simultaneous equations

$$\frac{x}{a} + \frac{z}{c} = \mu\left(1 - \frac{y}{b}\right);$$

$$\frac{x}{a} - \frac{z}{c} = \frac{1}{\mu}\left(1 + \frac{y}{b}\right).$$

The value of μ, found like that of λ, is

$$\mu = \frac{x_1}{a} + \frac{z_1}{c} \div 1 - \frac{y_1}{b} = 1 + \frac{y_1}{b} \div \frac{x_1}{a} - \frac{z_1}{c}.$$

Thus the equations of the second line become the same as

those already found for the first one, except that the signs of y_1 and y are changed. In part, we find

$$\frac{x_1x}{a^2} + \frac{y_1y}{b^2} - \frac{z_1z}{c^2} = 1;$$

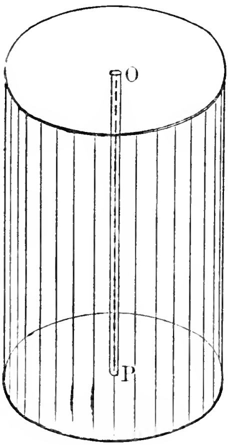
$$\frac{z_1x}{ac} - \frac{y}{b} - \frac{x_1z}{ac} = -\frac{y_1}{b}; \qquad (f)$$

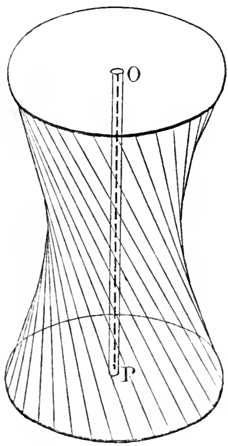
which are the equations of another line in the surface and passing through (x_1, y_1), thus proving the theorem.

264. The equations (e) and (f) represent two lines, each situated both in the surface and in the tangent plane. Hence the theorem may be expressed in the form:

THEOREM XIII. *Every tangent plane to the hyperboloid of one nappe intersects the surface in a pair of straight lines passing through the point of tangency.*

It is evident from the preceding theory that the surface in question may be generated by the motion of a line. We present three figures showing the relations which have been discussed. In the first, OP is a central axis or rod, supported on a fixed disk at the bottom and carrying

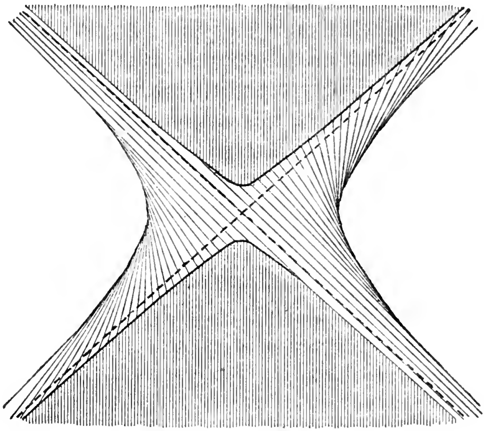




a similar disk at the top. The latter can be turned round on the rod. Vertical threads pass from all points of the circumference of one disk to the corresponding parts of the other, thus forming a cylindrical surface. Then turning the upper disk through any angle less than 180°, the threads will form an hyperboloid of revolution, as shown in the other

figure. The threads shown in the figure are those of one system only; by rotating the disk in the opposite direction the threads would be those of the other system.

The next figure represents the surface as cut by a plane very near the tangent plane, the section being an hyperbola of which the transverse



axis is vertical. By moving the cutting plane a little closer to the centre, the bounding curves of the section will merge into the dotted lines, and the plane will be a tangent to the surface at their point of intersection.

265. *Equations of the Generating Lines.* To study the lines in question, let us refer each to the point in which it intersects the plane of XY. We shall then have $z_1 = 0$ and

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1.$$

The equations of any one of the first set of lines will then become

$$\frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} = 1;$$

$$\frac{y}{b} - \frac{x_1 z}{ac} = \frac{y_1}{b}.$$

Because the lines lie in both of the planes represented by these equations taken singly, the coefficients l, m and n in the vectorial form must, by § 247, satisfy the conditions

$$\frac{x_1}{a^2}l + \frac{y_1}{b^2}m = 0;$$
$$\frac{m}{b} - \frac{x_1}{ac}n = 0.$$

These equations give the following values of l and m in terms of n, which remains arbitrary:

$$\left.\begin{aligned} l &= -\frac{ay_1}{bc}n; \\ m &= \frac{bx_1}{ac}n. \end{aligned}\right\} \text{First system of lines.}$$

Proceeding in the same way with the second set of lines, we find, starting from the equations (f),

$$\frac{x_1}{a^2}l + \frac{y_1}{b^2}m = 0;$$
$$\frac{m}{b} + \frac{x_1}{ac}n = 0;$$

from which

$$\left.\begin{aligned} l &= \frac{ay_1}{bc}n; \\ m &= -\frac{bx_1}{ac}n. \end{aligned}\right\} \text{Second system of lines.}$$

If we give n in both systems the value abc, so as to avoid fractions, the values of the direction-vectors l, m and n will be:

$$\text{First system:} \begin{cases} l = -a^2y_1; \\ m = +b^2x_1; \\ n = abc; \end{cases} \quad \text{Second system:} \begin{cases} l' = +a^2y_1; \\ m' = -b^2x_1; \\ n' = abc; \end{cases}$$

the values of the second system being distinguished by accents.

266. THEOREM XIV. *On an hyperboloid every line of the one system intersects all the lines of the other system. But no two lines of the same system intersect each other.*

Proof. Retaining $(x_1, y_1, 0)$ as the fundamental point of any line of the first system, and putting x_0 and y_0 for the

values of x_1 and y_1 in case of any line of the second system, the condition of intersection of two lines (§ 249) will be

$$(mn' - m'n)(x_1 - x_0) + (nl' - n'l)(y_1 - y_0) = 0, \qquad (a)$$

the third term being omitted because $z_1 = 0$ and $z_0 = 0$.

If we substitute for m, m', etc., their values, as just given, this equation becomes

$$ab^3c(x_1 + x_0)(x_1 - x_0) + a^3bc(y_1 + y_0)(y_1 - y_0) = 0.$$

Dividing by a^3b^3c, we find it reduce to

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - \left(\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2}\right) = 0.$$

Now, by hypothesis, (x_1, y_1) and (x_0, y_0) are points on the ellipse in which the plane of XY intersects the surface; that is, on the ellipse whose equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Hence the condition reduces to $1 - 1 = 0$, which is an identity, showing that the lines intersect.

Secondly. Let both lines belong to the same system, the one line intersecting the ellipse of the plane of XY in the point (x_1, y_1), as before, and the other in the point (x_0, y_0). We shall then have, for the values of the direction-vectors,

$$\begin{aligned} l &= -a^2y_1; & l' &= -a^2y_0; \\ m &= +b^2x_1; & m' &= +b^2x_0; \\ n &= abc; & n' &= abc. \end{aligned}$$

The condition (a) of intersection then becomes

$$\frac{(x_1 - x_0)^2}{a^2} + \frac{(y_1 - y_0)^2}{b^2} = 0.$$

Each term being a perfect square is necessarily positive, so that the condition of intersection is impossible. Q. E. D.

267. *Poles and Polar Planes of the Quadric.* Let us consider all possible tangent planes which pass through a fixed point (x_0, y_0, z_0) not belonging to the quadric. Let (x_1, y_1, z_1) be the variable point of tangency on the quadric.

The equation of the plane tangent at (x_1, y_1, z_1) will then be (13). The condition that this plane shall pass through the point (x_0, y_0, z_0) is that the co-ordinates of this point shall satisfy the equation of the plane, which gives

$$\frac{x_1 x_0}{a^2} + \frac{y_1 y_0}{b^2} + \frac{z_1 z_0}{c^2} - 1 = 0. \tag{14}$$

This is now a condition which the point of tangency (x_1, y_1, z_1) must satisfy as it varies. Being of the first degree, it shows that this point must lie in a certain plane. The equation of this plane may be written

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} + \frac{z_0 z}{c^2} - 1 = 0. \tag{15}$$

Def. That plane which contains the points of tangency of all tangents to a quadric which pass through a point is called the **polar plane** of that point.

The point is called the **pole** of the plane.

REMARK 1. The point of tangency in the above case may move along a curve which will then be the intersection of the polar plane and the quadric.

REMARK 2. The point (x_0, y_0, z_0) may be so situated that no real tangent plane can pass through it; for example, in the interior of an ellipsoid. The points of tangency (x_1, y_1, z_1) in (14) will then be entirely imaginary. But the plane (15) will always be real and determinate; only it will not meet the quadric. Hence:

THEOREM XV. *To every point in space corresponds a definite polar plane relative to any quadric.*

THEOREM XVI. Conversely, *To every plane corresponds a certain pole.*

Proof. Let the plane be

$$Ax + By + Cz + D = 0, \tag{a}$$

and let a, b and c be, as before, the principal axes of the quadric. Comparing the equation (a) with (15), we see that they become identical if we can have

$$\frac{x_0}{a^2} = -\frac{A}{D}; \qquad \frac{y_0}{b^2} = -\frac{B}{D}; \qquad \frac{z_0}{c^2} = -\frac{C}{D}$$

This only requires that we determine x_0, y_0 and z_0 by the conditions

$$x_0 = -\frac{a^2 A}{D}; \qquad y_0 = -\frac{b^2 B}{D}; \qquad z_0 = -\frac{c^2 C}{D};$$

which always give real values of x_0, y_0 and z_0, and therefore a real pole. Q. E. D.

Cor. If the plane approach the centre as a limit, D approaches zero as its limit, and x_0, y_0 and z_0 increase indefinitely. Hence

The pole of any diametral plane of a quadric is at infinity.

NOTATION. If we call any points P, Q, etc., we shall call their polar planes P', Q', etc.

268. THEOREM XVII. *If a point lie on a plane, the pole of the plane will lie on the polar plane of the point.*

Proof. Let a point P be (x_0, y_0, z_0) and a point Q be (x_1, y_1, z_1). Then, by (15), the polar plane Q' is

$$\frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} + \frac{z_1 z}{c^2} - 1 = 0.$$

Let the point P lie on this plane. Then the co-ordinates x_0, y_0, z_0 must satisfy this equation; that is,

$$\frac{x_1 x_0}{a^2} + \frac{y_1 y_0}{b^2} + \frac{z_1 z_0}{c^2} - 1 = 0.$$

This equation shows that the co-ordinates (x_1, y_1, z_1) satisfy the equation

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} + \frac{z_0 z}{c^2} - 1 = 0,$$

which is the equation of the polar plane of (x_0, y_0, z_0). Hence the pole (x_1, y_1, z_1), or Q, lies on this plane. Q. E. D.

Cor. If any number of points lie in a plane P', their polar planes will all pass through the pole P of that plane.

Conversely, *If any number of planes pass through a point Q, their poles will all lie on the polar plane of Q.*

THEOREM XVIII. *If any number of planes intersect in a straight line, their poles will all lie in another straight line.*

Proof. In order that planes may intersect in one line, it is necessary and sufficient that they should all pass through any two points, taken at pleasure, on that line. Let P and Q be two such points. Then—

Because all the polar planes pass through the point P, their poles all lie somewhere in the polar plane P';

Because these planes all pass through the point Q, their poles all lie somewhere in the polar plane Q'.

Hence these poles all lie on the intersection of P' and Q', which is a straight line. Q. E. D.

Cor. It is readily shown by reversing the course of reasoning that *if any number of points lie in a straight line, their polar planes will all pass through another line.*

Def. Two lines so related that all poles of planes passing through one lie in the other are called **reciprocal polars.**

EXERCISES

1. If an ellipsoid, an hyperboloid of one nappe and one of two nappes are formed with the same principal axes, a, b, c, it is required to write the equations of their several polar planes relatively to the pole (x_0, y_0, z_0).

2. In this case show that the polar planes with respect to the ellipsoid and the hyperboloid of one nappe intersect in the plane of XY on the line $b^2x_0x + a^2y_0y - 1 = 0$.

3. In the same case show that the polar planes with respect to the hyperboloids of one and of two nappes respectively are parallel.

4. In the same case show that the polar planes with respect to the ellipse and the hyperboloid of two nappes respectively intersect in a line parallel to the plane of XY and intersecting the axis of Z.

5. Show that if a pole lies on any diameter, the polar plane will be parallel to the diametral plane conjugate to such diameter.

6. Show that if the pole lie on the surface of the quadric, the polar plane will touch the surface at the pole.

7. Show that the reciprocal polar of a line tangent to a quadric is another line tangent at the same point, the two tangents lying in a pair of conjugate diametral planes.

8. If a line is required to lie in a diametral plane, show that its reciprocal polar must be parallel to the diameter conjugate to that plane.

Special Cases of Quadrics.

269. In all the preceding investigations it has been assumed that the co-ordinates A, B, C of the centre, given by the equations (2), and the quantities l', m', n' and d' in (5), which determine the three principal axes of the quadric, are all *finite* and *determinate.*

Although in the general case this will be true, yet the nine constants which determine the quadric may have such special values that these quantities may be zero or infinite. The complete discussion of these cases would require us to make extensive use of the theory of determinants, which we wish to avoid; we therefore shall merely point out to the student the possibility of certain special cases.

270. *The Paraboloid.* When we solve a system of three equations with three unknown quantities, like (2), each unknown quantity comes out as the quotient of two quantities (compare eq. (3) of § 188 for example), and the denominator of these quotients is the same for all the quantities. If this denominator approaches zero as a limit, the values of A, B and C in (2) will increase without limit. Hence, if this denominator vanishes, the centre of the quadric is at infinity.

In this case the quadric is called a **paraboloid.**

271. *The Cone.* In reducing the original equation to the form (3), the absolute term d' may vanish. In this case the principal axes a, b and c (§ 257) will all vanish (unless some of the quantities l', m' or n' are also zero), and we shall have the homogeneous equation

$$gx^2 + hy^2 + kz^2 + 2g'yz + 2h'zx + 2k'xy = 0. \qquad (16)$$

Def. A **cone** is a surface generated by the motion of a

line which passes through a fixed point and continually intersects a fixed curve.

The fixed point is called the **vertex** of the cone.

The fixed curve is called the **directrix** of the cone.

A **quadric cone** is one whose directrix is a plane locus of the second degree.

THEOREM XIX. *Every homogeneous equation of the second degree has for its locus a quadric cone whose vertex is at the origin.*

Taking the equation (16), which is a perfectly general one of the kind named in the theorem, we first prove that its locus is a cone having its vertex at the origin, in the following way:

We take any point at pleasure on the surface (16);

We pass a line through this point and through the origin;

We then show that this line must lie wholly on the locus.

Let (x_1, y_1, z_1) be any point on the surface (16). Then every point (x, y, z) determined by the equations

$$\left.\begin{aligned} x &= \rho x_1, \\ y &= \rho y_1, \\ z &= \rho z_1, \end{aligned}\right\} \qquad (a)$$

will lie on the line passing through the origin and (x_1, y_1, z_1). Substituting these values in (16) gives, for the condition that the point (x, y, z) shall lie on the locus,

$$\rho^2(gx_1^2 + hy_1^2 + kz_1^2 + 2g'y_1z_1 + 2h'z_1x_1 + 2k'x_1y_1) = 0. \quad (b)$$

By hypothesis (x_1, y_1, z_1) satisfies (16). Hence this condition (*b*) is satisfied *for all values of* ρ; hence every point determined by (*a*) lies on the surface (16); whence this surface is *some* cone.

Secondly, being of the second degree, (16) represents a quadric surface; whence, by Th. IV., every plane intersects it in a conic, and it is by definition a quadric cone.

REMARK. For the directrix of the cone we may take its intersection with any plane whatever not passing through the vertex. Let us then take the plane $z = c$. We then shall have from (16), for the equation of the directrix,

$$gx^2 + hy^2 + 2k'xy + 2h'cx + 2g'cy + kc^2 = 0. \qquad (17)$$

The coefficients being all independent, this curve may be any conic whatever. Hence (16) may represent any quadric cone whose vertex is at the origin.

272. *Special Case when a Quadric becomes a Pair of Planes.* Since the directrix of the cone may be any conic, it may be a pair of straight lines. Since a line turning on a point and intersecting a fixed line describes a plane, it follows that *whenever the directrix is a pair of lines, the quadric cone becomes a pair of planes.* Hence among the special kinds of quadrics must be included a pair of planes.

The quadric equation of a given pair of planes is readily found. If the equations of the planes are

$$\begin{aligned} ax + by + cz + d &= 0, \\ a'x + b'y + c'z + d' &= 0, \end{aligned}$$

we have only to take the product of these equations, which will be of the second degree in x, y and z.

273. *Surfaces of Revolution.* In the reduction of the general quadric, two of the principal axes, a and b for example, may be found equal. In this case the equation may be reduced to one of the forms

$$\frac{x^2 + y^2}{a^2} \pm \frac{z^2}{c^2} = \pm 1$$

or

$$x^2 + y^2 \pm a^2\left(\frac{z^2}{c^2} \pm 1\right) = 0. \qquad (17)$$

Assigning any constant value to z, the equation in x and y will be that of a circle. Hence all planes parallel to the plane of XY will intersect the quadric in circles having their centres on the axis of Z. Since all sections containing the axis of Z will be conics, the surface can be generated by the revolution of some conic around the axis of Z. It is therefore called a **surface of revolution.**

The equation (17) admits of the same four-fold classification as the equation (6), according to the algebraic signs of the ambiguous terms. We have therefore, as the three real forms—

I. *The ellipsoid of revolution:*

$$x^2 + y^2 = a^2\left(1 - \frac{z^2}{c^2}\right).$$

II. *The hyperboloid of revolution of one nappe:*

$$x^2 + y^2 = a^2\left(1 + \frac{z^2}{c^2}\right).$$

III. *The hyperboloid of revolution of two nappes:*

$$x^2 + y^2 = a^2\left(\frac{z^2}{c^2} - 1\right).$$

When, in the ellipsoid,

$c > a$, the ellipsoid is called **prolate;**
$c < a$, the ellipsoid is called **oblate;**
$c = a$, the ellipsoid is called a **sphere.**

In the hyperboloid of one nappe the axis c may be infinite. The equation will then be

$$x^2 + y^2 = a^2,$$

the equation of a cylinder of radius a, whose axis is that of Z.

274. *Deriving Surfaces from the Generating Curve.* The general method by which we find the equation of the surface generated by revolving a curve around the axis of Z is this:

Assume the curve to be in *any* initial position.

Take any point upon it whose vertical ordinate is z, and find the corresponding values of x and y, and hence of $\sqrt{x^2 + y^2}$, in terms of z, the distance of the point from the plane of XY.

Since this distance remains constant while the point revolves, the square of the equation thus found will be the equation of the surface.

If the fixed position can be so chosen that the generating curve may lie wholly in the plane of XZ (or YZ), one of the co-ordinates y or z will then be zero, and we have only to substitute $\sqrt{x^2 + y^2}$ for x or y, as the case may be.

275. *The Paraboloid of Revolution.* Let us suppose a parabola to revolve about its principal axis, which we shall take as the axis of Z. The square of each ordinate will then be $2pz$. But this square, as the curve revolves, is continually equal to $x^2 + y^2$, because the ordinate is a line perpendicular to the axis of Z, whose terminus on the curve is represented by the co-ordinates x and y of the curve. Hence the equation of the surface is

$$x^2 + y^2 = 2pz;$$

p being the semi-parameter of the generating parabola.

EXERCISES.

1. Find the equation of the cone generated by revolving around the axis of Z the straight line whose equation, when the line is in the plane XZ, is

$$x = mz + b.$$

Find also the vertex of the cone.

Ans. $x^2 + y^2 = m^2z^2 + 2mbz + b^2$.

Vertex at the point, $\left(0,\ 0,\ -\dfrac{b}{m}\right)$.

2. Investigate the surface generated by the motion of a straight line around an axis which does not intersect it, the shortest distance of the line from the axis being a, and the angle between them being α. (In the initial position we may suppose the line to intersect the axis of X at right angles at the distance a from the origin, and to form an angle α, whose tangent is m, with the axis of Z.) Find the equation of the surface and its principal axes.

Ans. $x^2 + y^2 - m^2z^2 = a^2$. Axes: $a,\ a,\ \dfrac{a}{m}$.

Here, if we take a point on the line at the distance r from the axis of X, its co-ordinates in the initial position will be

$$x = a; \quad z = r\cos\alpha; \quad y = r\sin\alpha = z\tan\alpha = mz.$$

3. Find the equation of the cone generated by the revolution of the straight line whose equations in one position are

$$x = m\rho; \qquad y = 0; \qquad z = n\rho.$$

Ans. $n^2(x^2 + y^2) - m^2z^2 = 0$.

4. Find the equation of the ellipsoid generated by the revolution of the ellipse $b^2x^2 + a^2z^2 = a^2b^2$.

5. If the hyperbola $c^2x^2 - a^2z^2 = a^2c^2$ revolve about the axis of Z, find the equation of the curve and of the cone described by the asymptotes.

$$\textit{Ans.}\quad \frac{x^2 + y^2}{a^2} - \frac{z^2}{c^2} = 1;$$

$$\frac{x^2 + y^2}{a^2} - \frac{z^2}{c^2} = 0.$$

6. Investigate the surface when the revolving hyperbola is $a^2z^2 - c^2x^2 = a^2b^2$.

7. Find the equation of the surface generated by the revolution about the axis of Z of the line whose equations are

$$x = a + l\rho;$$
$$y = b + m\rho;$$
$$z = c + n\rho.$$

$$\textit{Ans.}\quad n^2(x^2 + y^2) = (na - lc)^2 + (nb - mc)^2 + (l^2 + m^2)z^2 + 2(nla + mnb - l^2c - m^2c)z.$$

8. Show that the equation of a sphere of radius r whose centre is at the point (a, b, c) is

$$(x - a)^2 + (y - b)^2 + (z - c)^2 - r^2 = 0,$$

and find the value of r in order that the origin may bisect the radius passing through it.

9. Find the plane of the circle in which the spheres

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$$

and $$(x - a')^2 + (y - b')^2 + (z - c')^2 = r'^2$$

intersect each other.

10. Show that if three spheres mutually intersect each other, the planes of their three circles of intersection pass through a line perpendicular to the plane containing the centres. (One of the centres may be taken as the origin.)

11. Investigate the locus of the equation

$$\frac{x^2}{a} + \frac{y^2}{b} = z;$$

a and b being both positive.

12. Do the same thing for the equation

$$\frac{x^2}{a} - \frac{y^2}{b} = z.$$

13. Show that the six planes in which four circles taken two and two intersect each other all pass through a point.

14. Investigate the relations of the three surfaces

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1,$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$

and
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1.$$

Show that if these surfaces be cut by a plane parallel to that of XY, the two areas included between the three ellipses of intersection will each be constant and equal to the area of the ellipse in which the first surface intersects the plane of XY.

15. A straight line moves so that three fixed points upon it constantly lie in the three co-ordinate planes. Find the locus of a fourth point upon it whose distances from the other three points are a, b and c.

16. From the results of § 266 deduce the following conclusions:

I. The cosine of the angle between the two generating lines through the point $(x_1, y_1, 0)$ of the surface is

$$\frac{a^2b^2c^2 - a^4y_1^2 - b^4x_1^2}{a^2b^2c^2 + a^4y_1^2 + b^4x_1^2}.$$

II. At the ends of the respective axes a and b the cosines are $\dfrac{c^3 - b^2}{c^2 + b^2}$ and $\dfrac{c^2 - a^2}{c^2 + a^2}$.

III. If $a = b = c$, the lines are at right angles to each other.

PART III.

INTRODUCTION TO MODERN GEOMETRY.

276. *The Principle of Duality.* In modern geometry every straight line is supposed to extend out indefinitely in both directions, and is called a line simply. Hence lines, like points, differ only in situation.

Def. A **segment** is that portion of a line contained between two fixed points. Hence a segment is what is called a *finite straight line* in elementary geometry.

There are certain propositions relating to lines and points which remain true when we interchange the words *point* and *line*, provided that we suitably interpret the connecting words.

The principle in virtue of which this is true is called the **principle of duality.**

Two propositions which differ only in that the words *point* and *line* are interchanged are said to be **correlative** to each other.

The following are examples of correlative propositions and definitions. The right-hand column contains in each case the correlative of the proposition found at its left.

I. *Prop.* Through any *point* may pass an indefinite number of *lines.*	I. *Prop.* On any *line* may lie an indefinite number of *points.*
II. *Def.* Any number of lines passing through a point is called a **pencil of lines,** or simply a **pencil.**	II. *Def.* Any number of points lying on a line is called a **row of points**, or simply a **point-row** or **row.**

The common point of a pencil is called the **vertex** of the pencil.

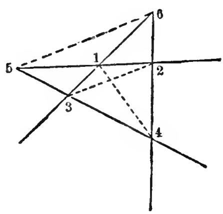
III. *Prop.* Two points determine the position of a certain line, namely, the line joining them.

IV. *Def.* The line joining two points is called the **junction-line** of the points.

V. *Prop.* Three points, taken two and two, determine by their junction-lines three lines.

VI. *Prop.* A collection of n lines, taken two and two, has, in general, $\frac{n(n-1)}{2}$ junction-points.

VII. *Def.* A collection of four lines, with their six junction-points, is called a **complete quadrilateral.**



VIII. *Prop.* On each of the four sides of a complete quadrilateral lie three vertices.

The line on which a row of points lie is called the **carrier** of the row.

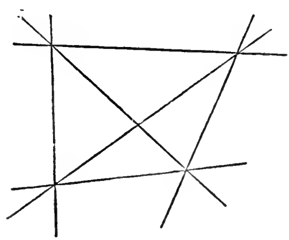
III. *Prop.* Two lines determine the position of a certain point, namely, their point of intersection.

IV. *Def.* The point of intersection of two lines is called the **junction-point** of the lines.

V. *Prop.* Three lines, taken two and two, determine by their intersections three junction-points.

VI. *Prop.* A collection of n points, taken two and two, has, in general, $\frac{n(n-1)}{2}$ junction-lines.

VII. *Def.* A collection of four points, with their six junction-lines, is called a **complete quadrangle.**



VIII. *Prop.* Through each of the four vertices of a complete quadrangle pass three sides.

IX. *Prop.* The complete quadrilateral has three *diagonals,* formed by joining the junction-point of each two sides to the junction-point of the remaining two sides.

If two lines are represented by the symbols a and b, their junction-point is represented by ab.

The pencil of lines from a vertex P to the points a, b, c, etc., is represented by P-abc, etc.

When two lines are each represented by a pair of point-symbols, a comma may be inserted between the pairs when their junction-point is expressed.

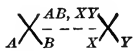
Ex. The expression ab, xy means the junction-point of the lines ab and xy.

IX. *Prop.* The complete quadrangle has three *minor vertices,* being the intersection of the junction-line of each two vertices with the junction-line of the remaining two.

If two points are represented by the symbols a and b, their junction-line is represented by ab.

The row of points in which a carrier R intersects the lines A, B, C, etc., is represented by R-ABC, etc.

When two points are each represented by a pair of line-symbols, a comma may be inserted between the pairs when their junction-line is expressed.



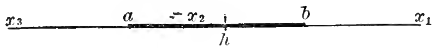
Ex. The expression AB, XY means the junction-line of the points AB and XY.

Scholium. When, in elementary geometry, two intersecting lines are drawn, their junction-point, being evident to the eye, is not separately marked. But when two points are given, it is considered necessary to draw their junction-line wherever this line is referred to. But this is not always necessary in the higher geometry, and such lines may be omitted when drawing them would make the figure too complicated.

The Distance-Ratio and its Correlative.

277. *The Distance-Ratio.* Heretofore the position of a point on a straight line has been expressed by its distance (positive or negative) from some other point, supposed fixed on the line.

The position of the point may also be expressed by the *ratio* of its distances from two fixed points on the line.



Let a and b be two fixed points on an indefinite line, which points we may regard as the ends of a segment ab of the line. Let x_1, x_2 and x_3 be three positions of a movable point x, and let us consider the ratio $ax : bx$ of the distances of x from the points a and b. If we put

h, the distance ab;
k, the distance ax;
r, the ratio $ax : bx$,

we shall have

$$r = \frac{k}{k-h}. \tag{1}$$

This fraction, or $ax : bx$, is called **the distance-ratio of the point x with respect to the points a and b.**

NOTATION. The distance-ratio is written

$$(a,\ b,\ x) \equiv \frac{ax}{bx}.$$

Let us now study the changes of value of the distance-ratio as the point moves along the line.

Assuming the positive direction to be toward the right, then, when x is in the position x_1, the distances ax_1 and bx_1 will both be positive, and we shall have

$$\left.\begin{array}{l} k > h; \\ r > +1. \end{array}\right\} \tag{2}$$

If x recedes indefinitely toward the right, k increases indefinitely and the ratio $\frac{k}{k-h}$ approaches unity as its limit. Therefore, for a point at infinity on the line, we have

$$r = +1.$$

Supposing the point to move toward the left, the denominator $k - h$ will become zero when x reaches b; and as this point is approached, the fraction r will increase without limit. Hence, when x is at b,

$$r = \infty.$$

When x is in the position x_2 between a and b, ax will be positive and bx negative. Hence, in this part of the line,

$$r = \text{a negative quantity}.$$

As x passes from b to a, r will increase from negative infinity to zero.

At a,

$$r = 0.$$

In the position x_3 both terms of the ratio will be negative, and r will be a positive proper fraction.

As x recedes to infinity on the left, r will approach unity as its limit. Hence, whether we suppose x to reach infinity in the negative or positive direction, we have, at infinity,

$$r = +1,$$

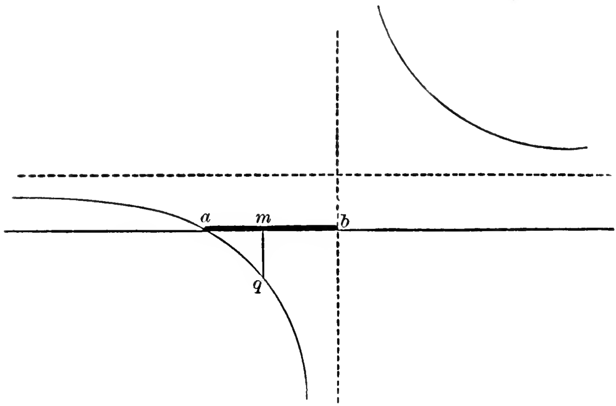
and no distinction is necessary between the two infinities.

If, then, we suppose the point x to move along the whole line from negative to positive infinity, we may consider it as arriving back at its starting-point, and being ready to repeat the motion. During this motion the distance-ratio r will also have gone through all possible values from negative to positive infinity, and will be back at its starting-point. The order of positions of the point and the order of changes of r are as follows:

Point: Infinity; negative; at point a; on line ab; at point b; positive; infinity.

Dist. r.: Unity; positive < 1; zero; negative; infinity; positive > 1; unity.

278. To exhibit to the eye the changes in r as x moves along the line, we may erect at each point of the line an ordinate the length of which shall represent the value of r at that point. The curve passing through the ends of the ordinates will be that required.



The ratio r being a pure number, the length which shall represent unity may be taken at pleasure. So we may lay down from the middle point m of ab an arbitrary length $mq \equiv -1$, and the lengths of all the other ordinates will be fixed.

279. THEOREM. *The position of a point is completely fixed by its distance-ratio with respect to two given points; that is, there can be only one point on the line to correspond to a given value of the distance-ratio.*

This is the same as saying that, in the equation (1), only one value of k will correspond to given values of r and h. This is readily proved by solving the equation with respect to k. We note that in the special case when $r = 1$ the point is at infinity.

280. *Relation of the Distance-Ratio to the Division of a Line.* The conception of the distance-ratio occurs in elementary geometry when we say that the point x divides the line ab internally or externally into the segments ax and bx, having

a certain ratio to each other. This ratio is identical with the distance-ratio just defined. It is *negative* when the line ab is cut *internally; positive* when it is cut *externally.*

We may therefore, instead of saying, "The distance-ratio of the point x with respect to the points a and b," say,

"The ratio in which the point x divides the segment ab."

EXERCISES.

1. Show that the curve which expresses the value of r in the preceding section is an hyperbola; find its asymptotes; define the class to which it belongs; construct its major axis in the case when we take $mq = ab$.

2. Show that if we take two points at equal distances on each side of the middle point m of the base-line, the product of the corresponding values of r will be unity. Translate this result into a property of the equilateral hyperbola.

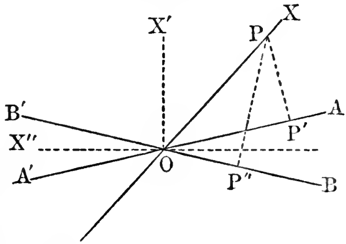
281. *The Sine-Ratio.* In the two preceding articles we showed how to express the position of a varying *point* upon a fixed *line*. The correlative of this problem is that of expressing the position of a varying *line* which must pass through a fixed *point*.

As, in the first case, the position of the moving point is expressed by its relation to two fixed points on the line, so, in the second case, the position of the moving line is fixed by its relation to two fixed lines passing through the point. Let us put

O, the fixed point;

OA, OB, the two fixed lines;

OX, the moving line.



From any point P of this line drop the perpendiculars PP' and PP'' upon the fixed lines. Let us then consider the ratio

$$R = \frac{PP'}{PP''}.$$

We readily see that the value of r is the same in whatever position on the line OX the point P is chosen, and that

$$R = \frac{\sin AOX}{\sin BOX}.$$

Hence we call R the **sine-ratio** *of the line OX with respect to the lines OA and OB.*

To investigate the algebraic signs of $\sin AOX$ and $\sin BOX$, let us take the directions OA, OB and OP as positive. Then, in accordance with the usual trigonometric convention, the sine of AOP will be *positive* or *negative* according as a person standing at O and facing toward A has the point P on the *left* or *right* side of the line OA.

Suppose the line OX to start from the position OA and to turn round O in the positive direction. Then,

As OX starts from OA,

R starts positively from zero.

When OX reaches the bisector OX',

$$R = +1,$$

because then $AOX + BOX = 180°$.

As OX approaches the position OB',

R increases indefinitely,

because $\sin BOX$ approaches zero.

When OX reaches OB',

$$R = \infty.$$

As OX passes from OB to the bisector OX'',

R increases from $-\infty$ to -1.

When OX reaches OX'',

$$R = -1.$$

As OX passes from OX'' to OA',

R increases from -1 to 0.

The line OX has now reached its initial position, though its positive direction is reversed. Completing the revolution,

we see that R goes through the same series of values as before. Hence

The sine-ratio depends only upon the position of the moving line, and is the same whether we take one direction or the other as positive.

282. *Division of the Angle.* As, in § 280, we have supposed the point x to divide the line ab, so we may in the preceding construction suppose the line OX to divide the angle BOA into the parts BOX and AOX. We then take for the dividing ratio, not the ratio of the angles themselves, but that of their sines.

NOTE. The student may remark a certain incongruity when we speak of the point x dividing the line ab into the segments ax and bx, because it is not the algebraic *sum* but the algebraic *difference* of the segments which makes up the line ab. This incongruity would be avoided by measuring one of the segments in the opposite direction, making x its initial point, thus taking ax and xb as the segments. But it is more convenient to take x as the terminal point of each segment, and to accept the incongruity of calling a line the algebraic difference of its parts, because no confusion will arise when the case is once understood.

The same remarks apply to the division of the angle.

283. *Distinction of Antecedent and Consequent in Distance- and Sine-Ratio.* In forming a ratio one of the terms must be taken as the *antecedent* (or dividend), and the other as the *consequent* (or divisor). By interchanging the points a and b the antecedents and consequents will be interchanged, and the ratio will therefore be changed to its reciprocal.

To give clearness to the subject we shall employ the following notation:

The points a and b from which we measure the segments ax and bx will be called *base-points.*

That base-point from which the antecedent segment of the ratio is measured will be called the *A-point.*

That base-point from which the consequent segment of the ratio is measured will be called the *B-point.*

Then, when the ratio $\frac{ax}{bx}$ is represented in the form

$$(a, b, x),$$

we write first the A-point, next the B-point, and lastly the terminal point, which we may call the *T-point*.

284. *Permutation of Points.* If we use the notation

$$p \equiv \text{length } ab,$$
$$q \equiv \text{length } bx,$$

we shall have, by the definition of the distance-ratio,

$$(a,\ b,\ x) = \frac{p+q}{q} = 1 + \frac{p}{q}. \qquad (a)$$

Let us represent this ratio by the symbol r. Then, by permuting the base-points between themselves (that is, by making a the B- and b the A-point), we shall have

$$(b,\ a,\ x) = \frac{q}{p+q} = \frac{1}{r}, \qquad (b)$$

a result which we may express by the general proposition:

I. *By permuting the base-points we change the distance-ratio into its reciprocal.*

By permuting b and x in (a), we have

$$(a,\ x,\ b) = \frac{ab}{xb} = \frac{p}{-q} = 1 - r. \qquad (c)$$

That is:

II. *By permuting the B- and T-points we change the distance-ratio r into $1 - r$.*

The same permutation applied to (b) gives

$$(b,\ x,\ a) = \frac{ba}{xa} = \frac{-p}{-(p+q)} = \frac{p}{p+q} = \frac{r-1}{r}. \qquad (d)$$

Lastly, by permuting the base-points in (c) and (d),

$$(x,\ a,\ b) = \frac{1}{1-r}; \qquad (e)$$

$$(x,\ b,\ a) = \frac{r}{r-1}. \qquad (f)$$

EXERCISES.

1. By comparing the forms (a) and (d), show that if, in the expression

$$\varphi r \equiv \frac{r-1}{r},$$

we put φr for r and repeat the substitution in the result, we shall get r itself.

2. Find the distance from the A-point (§ 277) of points whose distance-ratios are

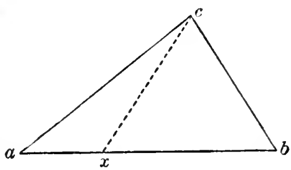
$$-2, -\frac{3}{2}, -\frac{1}{2}, +\frac{1}{2}, +\frac{3}{2}.$$

Ans. $\frac{2}{3}p$; $\frac{3}{5}p$; $\frac{1}{3}p$; $-p$; $3p$.

3. If a, b, x and y be any points whatever, show that

$$\frac{(a, b, x)}{(a, b, y)} = \frac{(b, a, y)}{(b, a, x)} = \frac{(x, y, a)}{(x, y, b)} = \frac{(y, x, b)}{(y, x, a)}.$$

4. Show that if from the vertex c of an isosceles triangle abc we draw a line cx to the base, the sine-ratio in which the angle c is divided by the line cx equals the distance-ratio in which the base ab is cut by the point x.



In algebraic language the theorem is

$$\frac{\sin acx}{\sin bcx} = \frac{ax}{bx}.$$

5. If the angle AOB is 120°, in what directions must those lines be drawn which will divide the angle in the respective sine-ratios -2 and $+2$?

6. If the point x divide the segment ab in the ratio $+1:2$, in what ratio will b divide the segment ax? *Ans.* $2:3$.

7. If the points x and y divide the segment ab in the respective ratios $+2$ and -2, in what ratios will a and b respectively divide the segment xy? *Ans.* $+3$ and -3.

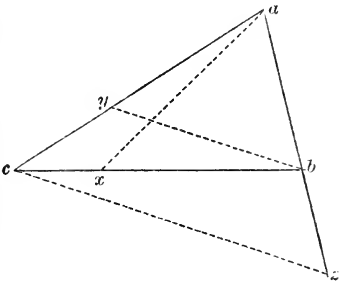
8. If the sum of the distance-ratios of two points, x and y, is unity, show that $ax \times ay = ab^2$.

Theorems involving the Distance- and Sine-Ratios.

285. *Def.* If each of the sides or angles of a polygon is divided by a point or line, the ratios of the divisions are said to be taken *in order* when each vertex is a divisor-point for one of its sides and a dividend-point for the other side.

If the divisions are all internal, we shall, in going round the polygon, have the divisor- and dividend-segments in alternation.

286. THEOREM I. *If any three lines be drawn from the three vertices of a triangle to its opposite sides, the continued product of the sine-ratios in which the angles are divided is equal to the continued product of the distance-ratios in which the sides are divided, the ratios being all taken in order.*



Hypothesis. A triangle abc of which the sides and angles are divided by the lines ax, by and cz.

Conclusion. If we put

r_1, r_2, r_3, the distance-ratios in which the sides are divided by the points x, y and z respectively;

R_1, R_2, R_3, the sine-ratios in which the angles are divided by the respective lines ax, by and cz, we have

$$r_1\, r_2\, r_3 = R_1\, R_2\, R_3.$$

Proof. By the theorem of sines in trigonometry we have, in the triangles bax and cax,

$$\frac{cx}{ax} = \frac{\sin cax}{\sin c};$$

$$\frac{bx}{ax} = \frac{\sin bax}{\sin b}.$$

Dividing the first equation by the second,

$$\frac{cx}{bx} = \frac{\sin cax}{\sin bax} \cdot \frac{\sin b}{\sin c}.$$

In the same way we find

$$\frac{ay}{cy} = \frac{\sin aby}{\sin cby} \cdot \frac{\sin c}{\sin a};$$

$$\frac{bz}{az} = \frac{\sin bcz}{\sin acz} \cdot \frac{\sin a}{\sin b}.$$

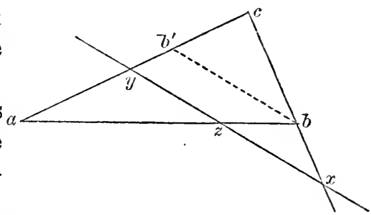
Taking the continued product of the three last equations, we have

$$\frac{cx}{bx} \cdot \frac{ay}{cy} \cdot \frac{bz}{az} = \frac{\sin cax}{\sin bax} \cdot \frac{\sin aby}{\sin cby} \cdot \frac{\sin bcz}{\sin acz}.$$

The three fractions in the first member of this equation are the distance-ratios in which the sides are divided, and those in the second member are the sine-ratios in which the angles are divided, so that the theorem is proved.

287. THEOREM II. *The continued product of the distance-ratios in which any transversal cuts the sides of a triangle is equal to unity.*

Proof. Let a transversal cut the sides of the triangle abc in the points x, y and z.



Through any vertex, as b, draw a line parallel to the transversal, meeting the opposite side in the point b'.

Then, forming the distance-ratios in which the sides bc and ba are divided, using the similar triangles thus constructed, we have

$$\frac{az}{bz} = \frac{ay}{b'y};$$

$$\frac{bx}{cx} = \frac{b'y}{cy};$$

while

$$\frac{cy}{ay} = \frac{cy}{ay}.$$

The continued product of these equations gives

$$\frac{az}{bz} \cdot \frac{bx}{cx} \cdot \frac{cy}{ay} = 1. \quad \text{Q. E. D.}$$

REMARK. Since the demonstration takes no account of algebraic signs, we have not yet shown whether the product is $+1$ or -1. It is evident that the transversal must cut either two sides of the triangle internally, or none. Hence either two factors or none at all will be negative; whence the product is always positive and equal to $+1$.

Corollary. If three points in a straight line be taken on the three sides of a triangle, the junction-lines from each point to the opposite vertex divide the angles into parts the continued product of whose sine-ratios is unity.

For, by Th. I., the product of the sine-ratios is equal to that of the distance-ratios, and, by Th. II., the continued product of the latter is unity.

288. THEOREM III. Conversely, *If on the three sides of a triangle abc we take any three points x, y, z, such that*

$$\frac{az}{bz} \cdot \frac{bx}{cx} \cdot \frac{cy}{ay} = 1,$$

these points will be in a straight line.

Proof. Let z' be the point in which the line xy cuts the side ab of the triangle. We shall then have, by Th. I.,

$$\frac{az'}{bz'} \cdot \frac{bx}{cx} \cdot \frac{cy}{ay} = 1.$$

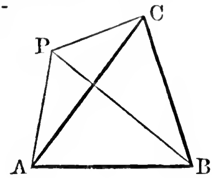
Comparing with the equation of the above hypothesis, we find

$$\frac{az}{bz} = \frac{az'}{bz'};$$

that is, the distance-ratios of the points z and z' with respect to a and b are the same.

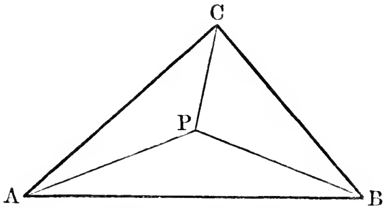
Because there is only one point on ab which has a given distance-ratio, the points z and z' are coincident and z lies on the line xy. Q. E. D.

289. THEOREM IV. *If three lines passing through a point be drawn from the vertices of a triangle, the angles will be so divided that the sine-ratios, taken in order, will be* -1.



Proof. If ABC be the triangle, and P the point, we have, in the triangles PAB, PBC and PCA, neglecting algebraic signs,

$$\frac{\sin BAP}{\sin ABP} = \frac{AP}{BP};$$
$$\frac{\sin CBP}{\sin BCP} = \frac{BP}{CP};$$
$$\frac{\sin ACP}{\sin CAP} = \frac{CP}{AP}.$$



The continued product of these equations would give

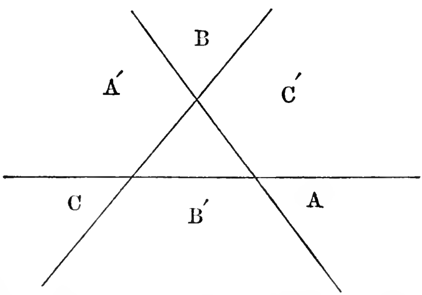
$$\frac{\sin BAP}{\sin CAP} \cdot \frac{\sin CBP}{\sin ABP} \cdot \frac{\sin ACP}{\sin BCP} = \pm 1. \qquad (a)$$

Algebraic signs having been neglected, it remains to be found whether this product is positive or negative. We have the theorems:

I. Lines drawn from any point within a triangle to the three vertices cut the angles internally.

II. Of the three lines drawn to the vertices of a triangle from an external point, and produced if necessary, two will divide the angles internally and one externally.

I is evident. To prove II let the whole plane without the triangle be divided by its sides into the six regions A, A', B, B', C and C'. Then the angle whose sides bound A will be cut internally or externally, according as the point is situated within or without one of the regions A and



A'. Considering the other angles in the same way, we see that only one angle can be cut internally and that the other two will be cut externally.

The sine-ratio being positive for an external and negative for an internal division, either one or all three of the factors in (a) must be negative. Hence

$$\frac{\sin BAP}{\sin CAP} \cdot \frac{\sin CBP}{\sin ABP} \cdot \frac{\sin ACP}{\sin BCP} = -1. \quad \text{Q. E. D.}$$

Corollary. Three lines passing from the vertices of a triangle through a point cut the opposite sides so that the continued product of the distance-ratios, taken in order, is negative unity.

For, by Theorem I., this product is equal to that of the corresponding sine-ratios, which product is negative unity, by the theorem.

290. THEOREM V. Conversely, *If three points cut the respective sides of a triangle so that the continued product of the distance-ratios is negative unity, the lines joining these points to the opposite vertices of the triangle pass through a point.*

Proof. If abc be the triangle, and x, y and z be the points, we have, by hypothesis,

$$\frac{az}{bz} \cdot \frac{bx}{cx} \cdot \frac{cy}{ay} = -1.$$

Join the points x and y to the opposite vertices, a and b, of the triangle by lines intersecting at a point O. From c draw a line L through O, and let z' be the point in which it cuts bc.

By Th. IV., *Cor.*, we then have

$$\frac{az'}{bz'} \cdot \frac{bx}{cx} \cdot \frac{cy}{ay} = -1.$$

Comparing with the hypothesis, we have

$$\frac{az}{bz} = \frac{az'}{bz'}.$$

Therefore the points z and z' are coincident and the line from z to c is identical with L, and so passes through the point O in which ax and by intersect. Q. E. D.

EXERCISES.

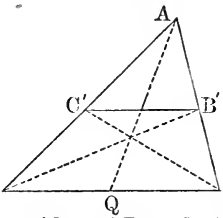
1. Explain what Theorem II. shows when the transversal is parallel to one of the sides.

2. What does Theorem IV. become when the point through which the lines are drawn is at infinity?

3. Show by Theorem V. that the three medial lines of a triangle pass through a point.

4. Show by the preceding theorems what bisectors of the interior and exterior angles of a triangle meet in a point.

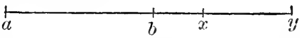
5. If from the vertices at the ends of the base BC of a triangle we draw lines intersecting on the medial line AQ and meeting the opposite sides in the points B' and C', show that $B'C'$ is parallel to BC.



6. In this case what relation exists between the distance-ratios in which the sides AB and AC are divided by the points B' and C'?

The Anharmonic Ratio.

291. Taking any point x on the line ab, we have, by what precedes, a distance-ratio $ax : bx$ or (a, b, x) of the point x with respect to the points a and b. In the same way, taking a fourth point y, we have a distance-ratio (a, b, y). Then:



Def. The quotient $\frac{(a, b, x)}{(a, b, y)}$ of the distance-ratios of the points x and y with respect to the points a and b is called the **anharmonic ratio** of the four points a, b, x and y.

That terminal point x which enters into the numerator of the fraction will be called the A-T-point; the other, the B-T-point.

It will be seen that the anharmonic ratio is a pure number whose value depends upon the mutual distances of the four points.

292. The following are simple corollaries from the definition of the anharmonic ratio:

I. *If the terminal points are both outside the segment ab, or both within it, the anharmonic ratio is positive.*

For in the first case the distance-ratios are both positive, and in the second they are both negative.

II. *If one terminal point is within and the other without the segment ab, the anharmonic ratio is negative.*

For the two distance-ratios then have opposite signs.

III. *If the two terminal points coincide, the anharmonic ratio is unity.*

IV. *If three points, namely, the base-points and one terminal point, are fixed, while the other terminal point may move, then for every value which we may assign to the anharmonic ratio there will be one and only one position of the movable point.*

For if we put $r \equiv$ the anharmonic ratio, and suppose the points a, b and x to be fixed, we have, by definition,

$$r = \frac{(a,\ b,\ y)}{(a,\ b,\ x)};$$

whence

$$(a,\ b,\ y) = (a,\ b,\ x) \times r.$$

Now, the points a, b and x being fixed, the quantity $(a,\ b,\ x)$ is a constant, so that for every different value we assign to r we shall have a different value of the distance-ratio $(a,\ b,\ y)$, and hence a different position of the point y (§ 279).

It will be seen that the four points which enter into an anharmonic ratio form two pairs, one pair being the two base-points, the other pair the two terminal points. The two points of each pair are said to be **conjugate** to each other.

NOTATION. We represent the anharmonic ratio $\dfrac{(a,\ b,\ x)}{(a,\ b,\ y)}$ in the form

$$(a,\ b,\ x,\ y).$$

Expressing the points by their general designations (§§ 283, 291), the order of writing them is

$$(A,\ B,\ A\text{-}T,\ B\text{-}T).$$

Writing the ratios at length, we have

$$(a, b, x, y) \equiv \frac{ax : bx}{ay : by} = \frac{ax \;.\; by}{ay \;.\; bx}. \qquad (a)$$

293. *Permutation of Points.* Let us now consider the problem, What changes will result in the anharmonic ratio by interchanging the different points?

By interchanging the two base-points, that is, by making b the A-point and a the B-point, we change each distance-ratio into its reciprocal (§ 284, I), and hence the anharmonic ratio into its reciprocal, because we always have

$$\frac{1 : p}{1 : q} = \frac{q}{p},$$

whatever be p and q.

The same result follows by interchanging the terminal points, because we then change the terms of the fraction $\frac{(a, b, x)}{(a, b, y)}$ into $\frac{(a, b, y)}{(a, b, x)}$.

Hence, if we make both changes, the anharmonic ratio will be restored to its original value.

If we simply make the base-points the terminal ones, and *vice versa*, the anharmonic ratio is unaltered. For, by the notation,

$$(x, y, a, b) = \frac{xa : ya}{xb : yb} = \frac{xa : xb}{ya : yb},$$

which is identical with (a), the signs of each of the four segments being changed.

It follows from this that there will be four permutations which will leave the anharmonic ratio unchanged, and four others which only change it to its reciprocal. They are as follows:

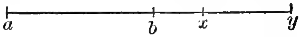
$$(a, b, x, y) = (b, a, y, x) = (x, y, a, b) = (y, x, b, a) \equiv r; \quad (1)$$

$$(b, a, x, y) = (a, b, y, x) = (x, y, b, a) = (y, x, a, b) = \frac{1}{r}. \quad (2)$$

In all these permutations the four points are paired in the same way, a and b being one pair and x and y the other.

Hence *the eight permutations which do not change the pairing of the conjugate-points can only interchange the terms of the anharmonic ratio.**

When the pairing of the points is changed, a may have either x or y as its conjugate-point. To find the effect of these permutations, we start from the following identical equation which always subsists between the six segments terminated by the four points a, b, x and y. These segments are ab, ax, ay, bx, by and xy.



$$ax.by + ab.yx + ay.xb \equiv 0. \tag{a}$$

To prove this equation, we substitute for ab and ay their values

$$ab = ax + xb,$$
$$ay = ax + xy,$$

and so write the first member of the equation in the form

$$ax.by + \left.\begin{array}{r} ax \\ + xb \end{array}\right| \left.\begin{array}{r} yx + ax \\ + xy \end{array}\right| xb,$$

which is the same as

$$ax(by + yx + xb) + xb(yx + xy),$$

an expression which vanishes identically, because

$$by + yx + xb \equiv 0; \quad yx + xy \equiv 0.$$

Now divide (a) by $ay.bx$. We thus find

$$\frac{ax.by}{ay.bx} + \frac{ab.yx}{ay.bx} = 1;$$

that is,

$$(a,\ b,\ x,\ y) + (a,\ x,\ b,\ y) = 1. \tag{b}$$

Hence, using the same notation as before,

$$(a,\ x,\ b,\ y) = 1 - r.$$

---

* In this pairing process note the analogy of conjugate-points to partners at whist. There are eight arrangements of the players around the table which will not change the pairing, and there are three ways in which the players may choose partners.

We now have, in the same way as before,

$$(a, x, b, y) = (x, a, y, b) = (b, y, a, x) = (y, b, x, a) = 1 - r; \quad (3)$$

$$(x, a, b, y) = (a, x, y, b) = (y, b, a, x) = (b, y, x, a) = \frac{1}{1 - r}. \quad (4)$$

We have finally to consider the case in which a is paired with y. To pair a with y, we remark that the equation (b), being true whatever points we suppose a, b, x and y to represent, may be considered a brief expression of the theorem:

By interchanging the B- and A-T-points, we form a new anharmonic ratio which, added to the original one, makes unity.

Applying this theorem to the second expression in line (4), it gives

$$(a, x, y, b) + (a, y, x, b) = 1.$$

Hence $(a, y, x, b) = 1 - (a, x, y, b) = 1 - \dfrac{1}{1 - r} = \dfrac{r}{r - 1}$,

and

$$(a, y, x, b) = (y, a, b, x) = (b, x, y, a) = (x, b, a, y) = \frac{r}{r - 1}; \quad (5)$$

$$(y, a, x, b) = (a, y, b, x) = (x, b, y, a) = (b, x, a, y) = \frac{r - 1}{r}. \quad (6)$$

The equations (1) to (6) include all 24 permutations of a, b, x and y, which, however, give rise to only 6 different values of the anharmonic ratio, namely,

$$r, \qquad \frac{1}{r}, \qquad 1 - r, \qquad \frac{1}{1 - r}, \qquad \frac{r}{r - 1}, \qquad \frac{r - 1}{r}. \qquad (c)$$

294. The preceding operations lead to a curious algebraic result. Suppose that, instead of starting with the equation (1), we had started with any of the others, (6) for example. We could then have obtained expressions for the remaining 20 anharmonic ratios by performing the same operations on (6) which we have actually performed upon (1), only the expression $\dfrac{1}{1 - r}$ would have taken the place of r all the way through. But, if the process is correct, we should then arrive at the same expressions for the other 20 anharmonic ratios which

we have actually found. The same being true if we start from any other of the six equations, we conclude:

If, in the set of expressions (c), we substitute for r any one expression of the set, the values of the several expressions will be changed into each other in such a way that the set will remain unaltered except in its arrangement.

As an illustration, let us substitute the sixth expression, $\frac{r-1}{r}$, for r all the way through the set (c). Then, by reduction of the fractions,

$$r \text{ will be changed to } \frac{r-1}{r};$$

$$\frac{1}{r} \text{ will be changed to } \frac{r}{r-1};$$

$$1-r \text{ will be changed to } \frac{1}{r};$$

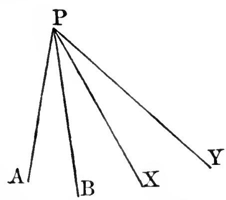
$$\frac{1}{1-r} \text{ will be changed to } r;$$

$$\frac{r}{r-1} \text{ will be changed to } 1-r;$$

$$\frac{r-1}{r} \text{ will be changed to } \frac{1}{1-r}.$$

We have thus reproduced the same set, only differently arranged.

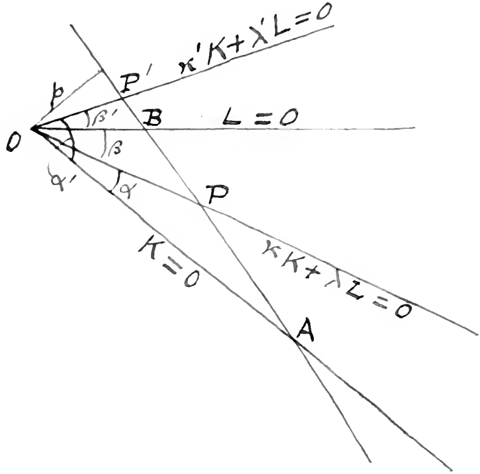
295. *Anharmonic Ratio of a Pencil of Lines.* As we have formed the anharmonic ratio of four points on a line by their distance-ratios, so we may form the anharmonic ratio of four lines passing through a point by means of their sine-ratios.



The four lines A, B, X and Y pass through the point P. If we take A and B as the base-lines forming the angle APB, the line X will give the sine-ratio

$$\frac{\sin APX}{\sin BPX},$$

Anharmonic Ratios.



$\angle AOP = \alpha$

$\angle AOP' = \alpha'$

$\angle BOP = \beta$

$\angle BOP' = \beta'$

$p\, AP = OA \cdot OP \sin\alpha \qquad p \cdot BP = OB \cdot OP \sin\beta$

$p\, AP' = OA \cdot OP' \sin\alpha' \qquad p \cdot BP' = OB \cdot OP' \sin\beta'$

$$\therefore \frac{p^2 AP \cdot BP'}{p^2 AP' \, BP} = \frac{\cancel{OA} \cdot \cancel{OP} \cdot \cancel{OB} \cdot \cancel{OP'} \sin\alpha \sin\beta'}{\cancel{OA} \cdot \cancel{OP'} \cdot \cancel{OB} \cdot \cancel{OP} \sin\alpha' \sin\beta}$$

$$\therefore \frac{AP}{AP'} : \frac{BP}{BP'} = \frac{\sin\alpha \sin\beta'}{\sin\alpha' \sin\beta}.$$

Let equations of

$OA \quad OB \quad OP \quad OP'$

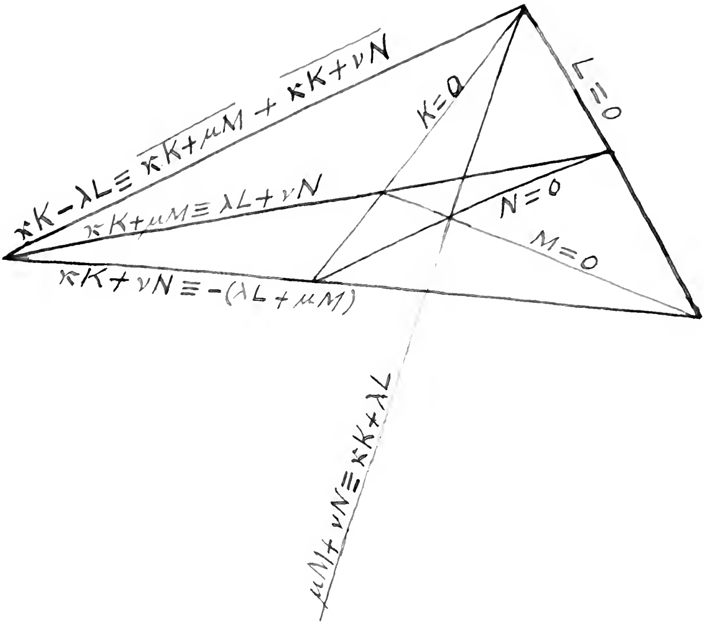
be $K = 0, \ L = 0 \quad \kappa K + \lambda L = 0, \ \kappa' K + \lambda' L = 0$

Then

$$\frac{K}{L} = -\frac{\lambda}{\kappa}, \qquad \frac{K}{L} = -\frac{\lambda'}{\kappa'}$$

$$-\frac{\lambda}{\kappa} = \frac{\sin\alpha}{\sin\beta}, \qquad -\frac{\lambda'}{\kappa'} = \frac{\sin\alpha'}{\sin\beta'}.$$

$$\therefore \frac{\lambda \kappa'}{\lambda' \kappa} = \frac{AP}{AP'} : \frac{BP}{BP'}.$$


$\kappa K - \lambda L \equiv \overline{\kappa K + \mu M} + \overline{\kappa K + \nu N}$
$K = 0$
$L = 0$
$\kappa K + \mu M \equiv \lambda L + \nu N$
$N = 0$
$M = 0$
$\kappa K + \nu N \equiv -(\lambda L + \mu M)$
$\mu M + \nu N \equiv \kappa K + \lambda L$

and the line Y will give the sine-ratio

$$\frac{\sin APY}{\sin BPY}.$$

The quotient of these ratios, or

$$\frac{\sin APX}{\sin BPX} : \frac{\sin APY}{\sin BPY} = \frac{\sin APX \sin BPY}{\sin BPX \sin APY},$$

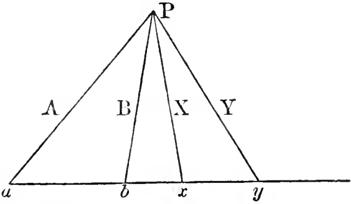
is called the **anharmonic ratio** of the pencil of lines PA, PB, PX and PY.

Designating each line by a single letter, we may write

$$(A,\ B,\ X,\ Y)$$

as the anharmonic ratio of the four lines A, B, X and Y.

296. FUNDAMENTAL THEOREM. *If a transversal cross a pencil of four lines, the anharmonic ratio of the four points of intersection will be equal to the anharmonic ratio of the pencil.*



Proof. Let $ABXY$ be the pencil, intersecting the transversal ay in the points a, b, x and y.

We begin, as in § 286, by comparing the distance-ratio (a, b, x) with the sine-ratio (A, B, X).

From the equations

$$ax : Px = \sin aPx : \sin xaP,$$
$$bx : Px = \sin bPx : \sin xbP,$$

we obtain, by division,

$$\frac{ax}{bx} = \frac{\sin aPx}{\sin bPx} \cdot \frac{\sin xbP}{\sin xaP};$$

or, using the abbreviated notation,

$$(a,\ b,\ x) = (A,\ B,\ X)\frac{\sin xbP}{\sin xaP}.$$

We find in the same way

$$(a,\ b,\ y) = (A,\ B,\ Y)\frac{\sin xbP}{\sin xaP}.$$

Taking the quotient of these equations,

$$\frac{(a,\ b,\ x)}{(a,\ b,\ y)} = \frac{(A,\ B,\ X)}{(A,\ B,\ Y)}.$$

The first member of this equation is, by definition, the anharmonic ratio of the four points a, b, x and y, while the second is that of the pencil of lines A, B, X and Y. Thence

$$(a,\ b,\ x,\ y) = (A,\ B,\ X,\ Y).$$

Q. E. D.

Cor. 1. *If any number of transversals cross the same pencil of four lines, the anharmonic ratios of the four points of intersection on the several transversals will all be equal.*

For each such ratio will be equal to the anharmonic ratio of the pencil.

Cor. 2. *If from a row of four points lines be drawn to a fifth movable point, the anharmonic ratio of the pencil thus formed will be constant, whatever be the position of the fifth point.*

For the anharmonic ratio of the pencil will be constantly equal to that of the row.

Scholium. Using the notation of § 276, IX., the preceding propositions may be expressed as follows:

If P be any vertex, and a, b, x, y a row of points, then

$$\text{Anh. ratio } (P\text{-}a,b,x,y) = (a,\ b,\ x,\ y).$$

If p be any transversal, and A, B, X, Y the four lines of a pencil, then

$$\text{Anh. ratio } (p\text{-}A,B,X,Y) = (A,\ B,\ X,\ Y).$$

297. *Application of the Principle of Duality.* The branch of Plane Geometry which we are now treating is subject to the principle of duality (§ 276); that is,

From every proposition respecting the relations of points and lines we may form a second correlative proposition respect-

ing the relation of lines and points, by interchanging the words line and point, as follows:

Line	instead of	Point.
Junction-point of two lines	"	Junction-line of two points.
Point on a line	"	Line through a point.
Pencil of lines	"	Row of points.
Three points in a line	"	Three lines through a point.
Anharmonic ratio of lines through a point	"	Anharmonic ratio of points on a line.

The correlative proposition is not necessarily different from the original one. When the two are identical, the proposition is *self-correlative.*

The relation of the proposition to its correlative is mutual; that is, the correlative of the correlative is the original proposition.

To make the notation correlative we represent the junction-point of the two lines A and B by AB.

Let us, as an example, change the preceding fundamental proposition into its correlative. The two then read:

Given: a $\left\{\begin{matrix}\text{pencil of four lines}\\ \text{row of four points}\end{matrix}\right.$ and any fifth $\left\{\begin{matrix}\text{line,}\\ \text{point,}\end{matrix}\right.$

We conclude: the anharmonic ratio of the four

junction-$\left\{\begin{matrix}\text{points}\\ \text{lines}\end{matrix}\right.$ of the four $\left\{\begin{matrix}\text{lines}\\ \text{points}\end{matrix}\right.$ with the fifth $\left\{\begin{matrix}\text{line}\\ \text{point}\end{matrix}\right.$ is equal to that of the given $\left\{\begin{matrix}\text{pencil.}\\ \text{row.}\end{matrix}\right.$

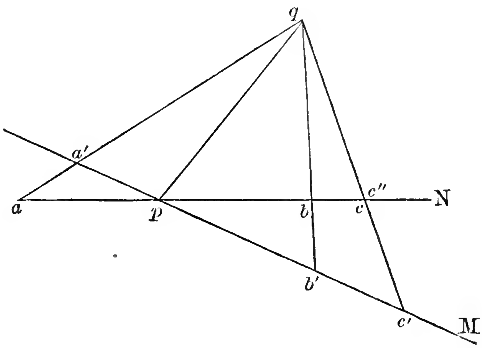
By reading the top lines we have the original proposition; by reading the bottom lines, its correlative. By making the construction it will be seen that the correlative proposition is identical with the original one.

The principle of duality applies to the demonstration as well as to the proposition. By making the above substitutions the demonstration of the original becomes the demon-

stration of the correlative. It is therefore in rigor not necessary to give the latter; and when we do so, it is only to assist the student.

298. THEOREM. *If we have two lines intersecting in a point p, and if we have on the one line any three points a, b, c, and on the other line three points a', b', c', such that the anharmonic ratio (p, a, b, c) is equal to (p, a', b', c'), then the three junction-lines aa', bb' and cc' meet in a point.*

Proof. Let M and N be the given lines, and let q be the junction-point of the lines aa' and bb'. Join qp and qc', and let c'' be the junction-point of the line N with the line qc'.



Then, because M and N are two transversals crossing the pencil q-a', p, b', c', we have

$$(p,\ a,\ b,\ c'') = (p,\ a',\ b',\ c').$$

By hypothesis,

$$(p,\ a,\ b,\ c) = (p,\ a',\ b',\ c').$$

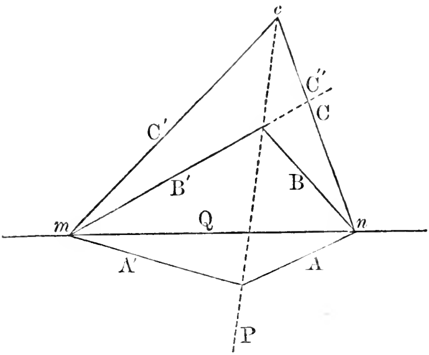
Hence

$$(p,\ a,\ b,\ c'') = (p,\ a,\ b,\ c).$$

The points p, a and b being given, there is only one fourth point which can form with them a given anharmonic ratio (§ 292, IV.). Hence the points c and c'' are coincident, and the junction-line $c'c$ is identical with $c'c''$, and so passes through the point q, in which, by construction, aa' and bb' intersect. Q. E. D.

Correlative Theorem. *If we have two points on a line Q, and if through one point pass three lines A, B and C, and through the other point pass three lines A', B' and C', such that the anharmonic ratio (Q, A, B, C) equals (Q, A', B', C'), then the three junction-points AA', BB' and CC' lie in a straight line.*

Proof. Let m and n be the points; let P be the junction-line of the points AA' and BB', c the junction-point PC', and C'' the junction-line nc.



Then, because the pencils Q, A', B', C' and Q, A, B, C'' pass through the same four points of the line P, we have

$$(Q, A, B, C'') = (Q, A', B', C').$$

But, by hypothesis,

$$(Q, A', B', C') = (Q, A, B, C).$$

Hence

$$(Q, A, B, C'') = (Q, A, B, C)$$

These equal anharmonic ratios having three lines identical, the fourth lines C and C'' are also identical; whence the lines C and C' intersect at c, the junction-point $C'P$; whence the junction-points AA', BB' and CC' all lie on the line P.

Q. E. D.

Note. The student should compare the demonstration, step by step, with that of the original proposition, and note the relation of each step.

Projective Properties of Figures.

299. Let there be a *point* O, a *plane* P and a *figure* Q each situated in any position in space. If lines (called lines of projection) pass from O to each point of Q, the points in which these lines intersect the plane P form a second figure which is called the **projection** of the figure Q.

This definition of a projection is more general than that of elementary geometry, in which the lines of projection are all parallel to each other and perpendicular to the plane P. The latter is a special case in which the point O is at infinity in a direction perpendicular to the plane.

It may be remarked that the shadow of a figure upon a plane, as cast by a luminous point, is identical with its projection. But should the distance of any part of the figure from the plane exceed the distance of the luminous point, there could be no shadow, but there would still be a projection, formed by continuing the lines of the rays in the reverse direction, namely, from the figure through the luminous point.

300. The following are some simple relations between figures and their projections:

I. *The projection of a point is a point.*

II. *The projection of a straight line is a straight line.*

For since the straight line and point lie in a plane, the lines of projection are all in this plane, and the projection is the intersection of this plane with the plane of projection.

III. *The projection of a row of points is another row whose carrier is the projection of the original carrier.*

IV. *The projection of a pencil is a pencil.*

V. *The projection of a curve and a tangent is another curve and a tangent.*

VI. *Every projection of a line passes through the point in which the line intersects the plane of projection.*

VII. *The projection of a circle is a conic section.*

For the lines from a point to the circumference of a circle form the elements of a cone. Hence their intersection with the plane of projection is the intersection of a conical surface with that plane, and is therefore, by definition, a conic section.

VIII. *If the projected figure Q is in a plane P', and if we call Q' its projection on the plane P, then Q itself is the projection of Q' upon the plane P'.*

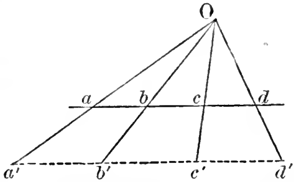
This follows at once from the definition, the lines of projection being identical in the two cases.

IX. *Every section of a circular cone can be projected into a circle.*

For, by taking the vertex of the cone as the point O, and its circular base as the plane of projection, the outline of this base becomes the projection of any section of the cone.

301. THEOREM. *The projection of a row of four points has the same anharmonic ratio as the original row.*

Proof. The lines of projection of the four points form, by definition, a pencil having its vertex at O. The carriers, both of the original and the projected row, form transversals crossing this pencil, and the two rows of points are the intersections of these carriers with the lines of the pencil. The anharmonic ratios of the two rows are therefore equal (§ 296, *Cor.* 1).

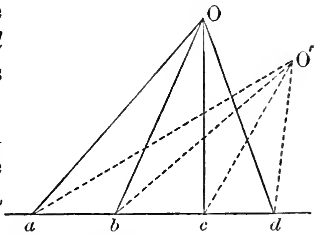


Q. E. D.

302. THEOREM. *The projection of a pencil of four lines has the same anharmonic ratio as the original pencil.*

Proof. Let O-$abcd$ be the given pencil, and let a, b, c and d be the points in which it intersects the plane of projection.

Because the lines of the pencil must all lie in one plane, the points a, b, c and d will lie in a straight line.



If O' be the projection of O, the projected pencil will be O'-$abcd$.

Then the anharmonic ratios of each pencil will be equal to (a, b, c, d), and so will be equal to each other.

REMARK. Those properties and relations of a figure which remain unchanged by projection are called *projective* properties and relations.

Harmonic Points and Pencils.

303. *Def.* When the anharmonic ratio of four points is negative unity, they are called a row of **four harmonic points,** and each pair of conjugate points is said to divide the segment joining the other pair **harmonically.**

So a pencil of four lines of which the anharmonic ratio is negative unity is called an **harmonic pencil.**

Cor. The anharmonic ratio being negative, one of the terminal points must divide the base-line internally and the other must divide it externally. Hence the order of the four points is such that the conjugate points of the one pair, a and b, alternate with those of the other pair, x and y.

If the point x is half way between a and b, its conjugate, y, is at infinity.

If x then move toward b, y will also move toward b from the right, and the two points will reach b together.

If x move toward a, y will approach from infinity on the left, and the two points will reach a together.

The law of change is expressed by the following theorem:

304. THEOREM. *The product of the distances of two terminal harmonic points from the middle of the base-line is constant, and equal to the square of half the base-line.*

Proof. The condition that the anharmonic ratio of the four points a, b, x, $y = -1$ is

$$\frac{ax}{bx} : \frac{ay}{by} = -1,$$

which is equivalent to

$$ax.by + ay.bx = 0.$$

Let m be the middle point of ab. Then

$$\begin{aligned} ax &= am + mx; \\ by &= bm + my = -am + my; \\ ay &= am + my; \\ bx &= -am + mx. \end{aligned}$$

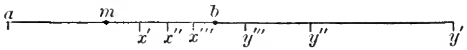
By substitution the equation (a) reduces to

$$-2(am)^2 + 2mx.my = 0.$$

Hence

$$mx.my = (am)^2. \qquad (1) \quad \text{Q. E. D.}$$

REMARK. On the line ab we may take any number of



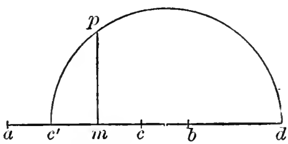
pairs of points, x and y, fulfilling the condition (1), and therefore dividing harmonically the segment ab.

Def. Three pairs of points which divide harmonically the same segment are said to form an **involution.**

305. *The Fourth Harmonic.*

Def. When three points of an harmonic row are given, the fourth is called the **fourth harmonic** of the other three.

PROBLEM. *Having given three points of an harmonic row, to find the fourth.*



Construction. Let a, b and c be the given points, and let a and b be the conjugate base-points.

On the middle point m of ab erect a perpendicular $mp = \frac{1}{2}ab$, and on the other side of m from c take $mc' = mc$.

Through p and c' describe a circle having its centre upon the line ab.

The other point, d, in which this circle cuts ab will be the fourth harmonic required.

Proof. From eq. (1) and from the theorem of elementary geometry which gives $c'm.md = (mp)^2$ the proof is readily found.

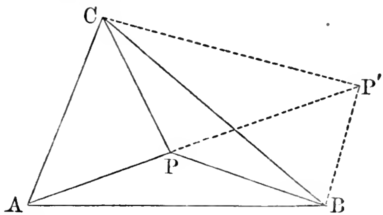
306. *Fourth Harmonic of a Pencil.* When three lines of a pencil are given, the fourth line necessary to form an harmonic pencil may be constructed.

From § 296 it follows that every pencil of four lines passing through a row of four harmonic points is an harmonic pencil, and, conversely, that every transversal intersects an harmonic pencil in four harmonic points.

Hence, to construct the fourth harmonic of a given pencil of three lines, we draw any transversal, find the fourth harmonic of the three points of intersection, and join it to the vertex.

307. *Harmonic Properties of the Triangle.*

THEOREM. *If from a point are drawn three lines to the vertices of a triangle, and at any two of the vertices the fourth harmonics to the lines thence emanating are constructed, these fourth harmonics will meet the line from the third vertex in a point.*



Proof. Let ABC be the triangle, and P the point. Let CP' and BP' be the fourth harmonics of the pencils $C\text{-}APB$ and $B\text{-}APC$.

Then, because AP, BP and CP meet in a point,

$$\frac{\sin BAP}{\sin CAP} \cdot \frac{\sin CBP}{\sin ABP} \cdot \frac{\sin ACP}{\sin BCP} = -1. \qquad (\S\ 289)$$

Because BP' and CP' are fourth harmonics conjugate to BP and CP respectively,

$$\frac{\sin CBP}{\sin ABP} = -\frac{\sin CBP'}{\sin ABP'};$$

$$\frac{\sin ACP}{\sin BCP} = -\frac{\sin ACP'}{\sin BCP'}.$$

By substitution in the equation (a) we have

$$\frac{\sin BAP}{\sin CAP} \cdot \frac{\sin CBP'}{\sin ABP'} \cdot \frac{\sin ACP'}{\sin BCP'} = -1.$$

Therefore the three lines AP, BP' and CP' meet in a point (§§ 286, 290). Q. E. D.

Scholium. By drawing the fourth harmonic at A and considering the other two pairs of vertices, A, B and C, A, we have two other points of meeting, making four in all.

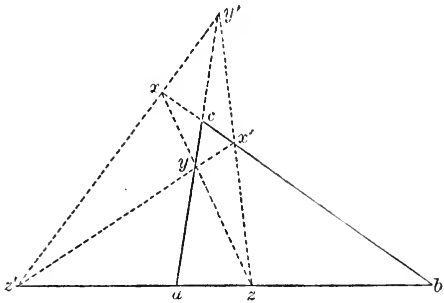
If the point P is the centre of the inscribed circle, the lines from P will be the bisectors of the angles, and the three points P' will be the centres of the escribed circles.

308. Correlative Theorem. *If three points on a line be taken on the sides of a triangle, and the fourth harmonics to two of them be constructed, these fourth harmonics will be on a line with the third point.*

Proof. Let abc be the triangle;

x, y and z, three points on the sides in a right line;

x', y' and z', the fourth harmonics to the rows b, c, x; c, a, y; a, b, z, respectively.



Because x', y' and z' are fourth harmonics,

$$\left.\begin{aligned} \frac{az}{bz} &= -\frac{az'}{bz'}; \\ \frac{bx}{cx} &= -\frac{bx'}{cx'}; \\ \frac{cy}{ay} &= -\frac{cy'}{ay'}. \end{aligned}\right\} \qquad (b)$$

Because x, y and z are on a right line,

$$\frac{az}{bz} \cdot \frac{bx}{cx} \cdot \frac{cy}{ay} = 1.$$

By substituting for any two of these factors their values from (b), we prove the theorem, by § 288.

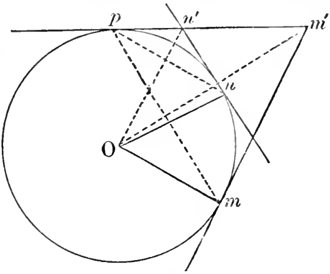
EXERCISES.

1. Any two orthogonal circles cut the line joining their centres in four harmonic points.

2. Any circle of a family having a common radial axis cuts harmonically the common chord of the orthogonal family.

Anharmonic Properties of Conics.

309. LEMMA. *If two tangents to a circle are intersected by a third tangent, the points of intersection subtend from the centre of the circle an angle measured by one half the arc between the two tangents.*



Proof. Let O be the centre of the circle;

m, n, the points of tangency of the two tangents;

p, the third point of tangency;

m', n', the points of intersection.

It is then easily shown, from the equality of the lines $m'p$ and $m'm$, that Om' is perpendicular to pm.

In the same way, On' is perpendicular to pn;

$$\therefore \qquad \text{Angle } m'On' = \text{angle } mpn.$$

By a fundamental property of the circle,

$$\text{Angle } mpn = \tfrac{1}{2} \text{ angle } mOn.$$

Therefore

$$\text{Angle } m'On' = \tfrac{1}{2} \text{ angle } mOn.$$

Q. E. D.

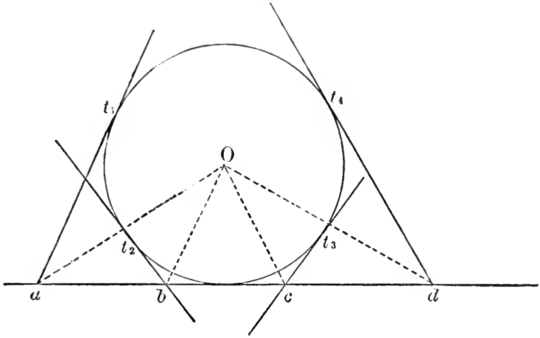
Cor. If the third tangent pn' moves around the circle, the angle $m'On'$ will remain constant, being always equal to $\frac{1}{2}mOn$.

310. THEOREM. *If four fixed tangents touch a conic, and a movable fifth tangent intersect them, the anharmonic ratio of the four points of intersection is the same for all positions of the fifth tangent.*

Proof. Let the conic be projected into the circle whose centre is O, and let a, b, c and d be four points in which the fifth tangent intersects four fixed projected tangents, touching the circle at the points t_1, t_2, t_3, t_4.

$$\left.\begin{array}{ll} \text{Because} & \text{Angle } aOb = \tfrac{1}{2} \text{ arc-angle } t_1t_2, \\ & \text{Angle } bOc = \tfrac{1}{2} \text{ arc-angle } t_2t_3 \\ \text{and} & \text{Angle } cOd = \tfrac{1}{2} \text{ arc-angle } t_3t_4, \end{array}\right\} \qquad (\S\, 309)$$

and t_1, t_2, etc., are, by hypothesis, fixed, these angles remain constant however the fifth tangent may move.

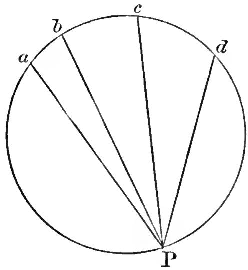


Therefore the anharmonic ratio of the pencil *O-abcd* remains constant, being a function of the sines of constant angles.

Therefore the anharmonic ratio of the row *a, b, c, d* remains constant (§ 296).

Because this anharmonic ratio is constantly equal to that of the corresponding points in the projected figure (§ 301), the latter also remains constant. Q. E. D.

311. Theorem. *If from four fixed points of a conic lines be drawn to a fifth variable point, the anharmonic ratio of the pencil thus formed will remain constant whatever the position of the fifth point.*



Proof. Project the conic into a circle. Let *a, b, c* and *d* be the projections of the four fixed points, and *P* that of the fifth point.

By a fundamental property of the circle the angles *aPb, bPc,* etc., will remain constant as *P* moves on the circle.

Therefore the anharmonic ratio of the pencil *P-abcd* will remain constant.

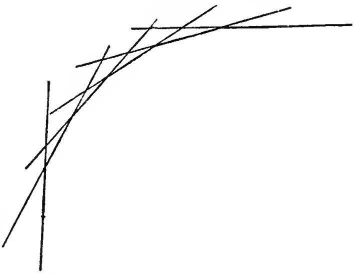
Therefore the anharmonic ratio of the corresponding pencil in the conic will remain constant (§ 302). Q. E. D.

Def. The constant anharmonic ratio of a pencil whose

lines pass from four fixed points on a conic to any fifth point is called the *anharmonic ratio of the four points of the conic.*

Def. The constant anharmonic ratio of the points in which four fixed tangents to a conic intersect a fifth tangent is called the *anharmonic ratio of the four tangents to the conic.*

312. *Extension of the Principle of Duality to Curves.* If we conceive a series of points to follow each other according to some law, their junction-points will form a broken line or a polygon. If each point of the series approaches indefinitely near to the preceding one, the broken line approaches a curve as its limit. We may therefore define a curve as the limit of a series of junction-lines when the points approach each other indefinitely.



The correlative conception, on the principle of duality, will be that of a series of lines following each other according to some law, and approaching each other indefinitely. The junction-points of consecutive lines will be the correlatives of the broken lines, and as they approach each other indefinitely they will tend to lie on some curve as their limit.

In the first case, if we suppose the points to be consecutive positions of a moving point, this point will move on the limiting curve.

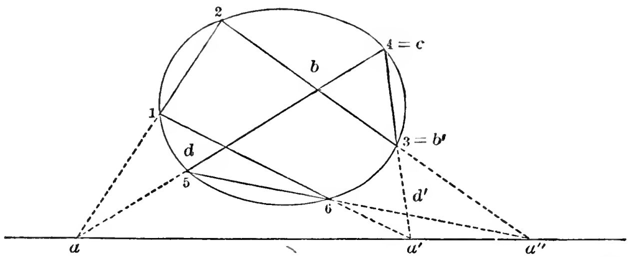
In the correlative case, if we suppose the lines to be the consecutive positions of a moving line, this line will constantly be tangent to the limiting curve.

We thus have, as correlative conceptions:

Points on a curve	corresponding to	Tangents to a curve.
Junction-point of two tangents	"	Junction-line of two points, i.e., a chord.
Points in which a line intersects a curve	"	Tangents from a point to the curve.

313. PASCAL'S THEOREM. *If a hexagon be inscribed in a conic, the three junction-points of its three pairs of opposite sides lie in a straight line.*

REMARK. By a polygon inscribed in a curve is meant any chain of straight lines, returning into itself, whose consecutive junction-points all lie on the curve. A polygon of n sides may be formed by taking any n points and joining them consecutively in any order whatever.



Proof. Let 1 2 3 4 5 6 be the inscribed hexagon, of which the opposite sides are

12	and	45;
23	and	56;
34	and	61.

Select three alternate vertices, as 2, 4 and 6, and consider the pencils

2-1345 and 6-1345.

Because these pencils are formed by joining the four fixed points 1, 3, 4, 5 on the conic to the points 2 and 6 respectively, their anharmonic ratios are equal (§ 311).

Let a, b, c, d be the row of points in which the pencil 2-1345 intersects the line 45. We shall then have

$$(a, b, c, d) = \text{Anh. ratio } (2\text{-}1345). \qquad (\S\ 296)$$

Let a', b', c, d' be the row of points in which the pencil 6-1345 intersects the line 34. We have

$$(a', b', c, d') = \text{Anh. ratio } (6\text{-}1345).$$

Hence

$$(a, b, c, d) = (a', b', c, d').$$

These two rows have the point c common to both, and this point occupies the same position in the two ratios. Therefore the three lines

$$aa', \; bb', \; dd'$$

meet in a point (§ 298).

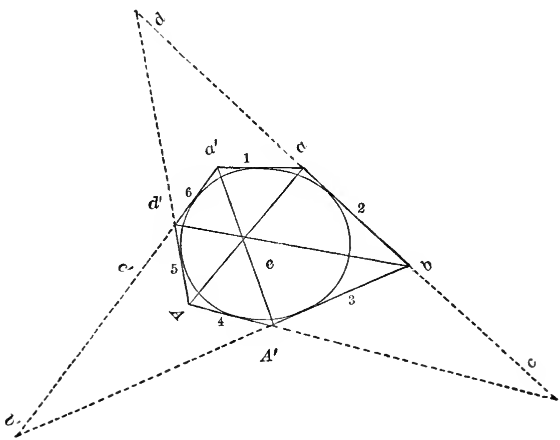
But a and a' are the respective junction-points (1 2 and 4 5) and (6 1 and 3 4), while $bb' \equiv 2\,3$ and $dd' \equiv 5\,6$.

Hence the opposite sides 2 3 and 5 6 intersect on the line aa' which joins the junction-points of the two other pairs of opposite sides. Q. E. D.

314. Correlative of Pascal's Theorem: Brianchon's Theorem. *The three lines joining the opposite vertices of a hexagon circumscribed about a conic meet in a point.*

Proof. Let the sides taken in order be 1, 2, 3, 4, 5, 6.

Consider the two rows of points a, b, c, d and a', b', c', d' in which the sides 1, 3, 4 and 5 intersect 2 and 6.



Because the tangents 2 and 6 are each intersected by the four tangents 1, 3, 4, 5, we have

$$(a, b, c, d) = (a', b', c', d'). \qquad (\S\,310)$$

Consider the pencil A-a,b,c,d. We have

$$\text{Anh. ratio } (A\text{-}a,b,c,d) = (a, b, c, d). \qquad (\S\,296)$$

Harmonic Proportion.

Relation of the algebraic and geometric expressions.

Let

ρ_2, ρ, ρ_1 = three successive terms of a harmonic progression

ρ = harmonic mean bet. ρ_2 and ρ_1

$\therefore \frac{1}{\rho_2}, \frac{1}{\rho}, \frac{1}{\rho_1}$ = three successive terms of an arithmetical progression.

Let d = common difference

Then

$$\frac{1}{\rho_2} = \frac{1}{\rho} + d, \quad \frac{1}{\rho} = \frac{1}{\rho_1} + d$$

$$\therefore \quad \frac{1}{\rho_2} - \frac{1}{\rho} = \frac{1}{\rho} - \frac{1}{\rho_1} \text{ and } \frac{\rho - \rho_2}{\rho\rho_2} = \frac{\rho_1 - \rho}{\rho\rho_1}$$

i.e.
$$\frac{\rho_2}{\rho - \rho_2} = \frac{\rho_1}{\rho_1 - \rho}.$$

Then if $\rho = OP$, $\rho_1 = OP_1$, $\rho_2 = OP_2$

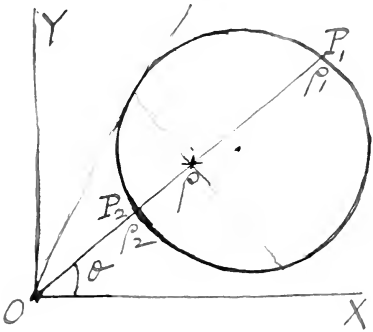
we have as the geometric expression of the harmonic proportion (disregarding signs)

$$\frac{P_2O}{P_2P} = \frac{P_1O}{P_1P}.$$

The algebraic expression for the harmonic mean is

$$\frac{1}{\rho} = \frac{1}{2}\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right).$$

The Harmonic Conjugates on the Secant of a Circle.



$$x = \rho\cos\theta$$
$$y = \rho\sin\theta$$

$$(x-a)^2 + (y-b)^2 - r^2 = 0$$

$$x^2 + y^2 \underbrace{- 2ax - 2ay}_{\parallel} + \underbrace{a^2 + b^2 - r^2}_{\parallel} = 0$$

$$\rho^2 - \quad 2m\rho \quad + \quad k \quad = 0$$

$$\rho = m \pm \sqrt{m^2 - k}$$

$$\rho_1 = m + \sqrt{m^2 - k}$$

$$\rho_2 = m - \sqrt{m^2 - k}$$

$$\frac{1}{\rho} = \frac{1}{2}\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right) = \frac{1}{2}\left(\frac{1}{m+\sqrt{m^2-k}} + \frac{1}{m-\sqrt{m^2-k}}\right) = \frac{m}{k}$$

$$\therefore \quad \rho m = k$$

or $\quad a\rho\cos\theta + b\rho\sin\theta - k = 0$

i.e. $\quad ax + by - a^2 - b^2 + r^2 = 0$

which is the equation of the polar of the origin O.

Also, for the same reason

$$\text{Anh. ratio } (A'\text{-}a',b',c',d') = (a', b', c', d').$$

Hence

$$\text{Anh. ratio } (A\text{-}a,b,c,d) = \text{Anh. ratio } (A'\ a', b', c', d').$$

Now these two pencils have the line $Ac \equiv A'c'$ common, and occupying the same position in the two ratios. Hence the three remaining lines intersect in three points in a right line (§ 298, Cor.); that is, the three points

(e) where diagonal Aa crosses diagonal $A'a'$;
(b) where line Ab crosses $A'b'$;
(d') where line Ad crosses $A'd'$,

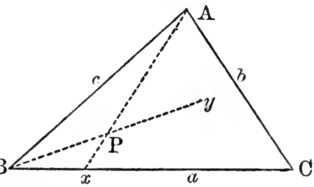
are in a right line, which proves the theorem.

Trilinear Co-ordinates.

315. In the method of trilinear co-ordinates the position of a point is defined by its relation to the three sides of a general triangle. Distances from each side are considered positive when measured in the direction of the opposite vertex; negative in the opposite case.

THEOREM. *If a general triangle be given, the position of a point is completely determined when the mutual ratios of its distances from the three sides of the triangle are given.*

Proof. Let it be given that the distances of a point P from the sides AB and AC of a triangle are in the ratio $m : n$.



If we draw a line Ax dividing the angle BAC in the sine-ratio $-\frac{m}{n}$, every point of this line will fulfil the given condition (§ 281).

If it be also given that the distances of the point P from the sides BC and BA are in the ratio $p : m$, then the point P must also lie on the line By dividing the angle B in the sine-ratio $-\frac{p}{m}$.

Hence, if both ratios be given, the only point which will satisfy them is the junction-point of the lines Ax and By, which is therefore the required point P. Q. E. D.

Method of Expressing the Ratios of Distances. The mutual ratios of the distances of a point P from the three sides of a triangle are most conveniently expressed by three numbers proportional to these distances. Let us put

d_1, d_2, d_3, the distances of P from the three sides of the triangle;

x_1, x_2, x_3, any three numbers proportional to d_1, d_2, d_3.

We then have

$$x_1 : x_2 : x_3 = d_1 : d_2 : d_3, \tag{1}$$

and
$$\frac{d_1}{d_2} = \frac{x_1}{x_2}; \qquad \frac{d_2}{d_3} = \frac{x_2}{x_3}; \qquad \frac{d_3}{d_1} = \frac{x_3}{x_1}; \tag{2}$$

also,
$$\frac{d_1}{x_1} = \frac{d_2}{x_2} = \frac{d_3}{x_3}. \tag{3}$$

If we put ρ for the common value of the three fractions (3), we have

$$\left.\begin{aligned} d_1 &= \rho x_1; \\ d_2 &= \rho x_2; \\ d_3 &= \rho x_3. \end{aligned}\right\} \tag{4}$$

The sets of equations (1), (2), (3) and (4) are so many different ways of expressing the fundamental fact of the proportionality of the numbers x_1, x_2 and x_3 to the distances d_1, d_2 and d_3.

316. *Relation between the Distances.* The position of the point is completely determined when its actual distances from any two sides of the triangle are given. Hence from two distances the third may be found, which shows that there is some equation between the distances. If we put

a, b, c, the lengths of the three sides of the fundamental triangle;

A, the area of the triangle,

the equation in question is

$$ad_1 + bd_2 + cd_3 = 2A. \tag{5}$$

This equation is readily proved by drawing lines from the point to the three vertices and equating the algebraic sum of the areas of the three triangles thus formed to that of the original triangle.

When x_1, x_2 and x_3 are given, this last equation with the three equations (4) suffice for the determination of d_1, d_2, d_3 and ρ, and therefore for the position of the point. In fact, by substituting (4) in (5), we have at once

$$\rho(ax_1 + bx_2 + cx_3) = 2A,$$

from which ρ is found. Then from (4) we have the values of d_1, d_2 and d_3.

317. *Multiplication by Constant Factors.* We may, instead of taking x_1, x_2 and x_3 proportional to d_1, d_2 and d_3, take them proportional to the products obtained by multiplying each distance d by any arbitrary but constant factor. If we take μ_1, μ_2 and μ_3 for such factors, we shall then have

$$x_1 : x_2 : x_3 = \mu_1 d_1 : \mu_2 d_2 : \mu_3 d_3,$$

or

$$\left.\begin{aligned} \mu_1 d_1 &= \rho x_1; \\ \mu_2 d_2 &= \rho x_2; \\ \mu_3 d_3 &= \rho x_3. \end{aligned}\right\} \tag{6}$$

The constant factors μ_1, μ_2 and μ_3 being supposed given, the equations (5) and (6) suffice for the determination of d_1, d_2, d_3 and ρ.

318. *Definition of Trilinear Co-ordinates.* The **trilinear co-ordinates** of a point are three numbers proportional to the distances of the point from the three sides of a triangle, each distance being, if we choose, first multiplied by any fixed factor.

The triangle from whose sides the distances are measured is called the **fundamental triangle.**

Corollaries from the Definition :

I. *If the trilinear co-ordinates of a point be all multiplied by the same factor, the position of the point which they represent will not be altered.*

For the position of the point depends only on the mutual

ratios of its trilinear co-ordinates, which remain unaltered by such multiplication.

II. *The points* (1), (2) *and* (3) *whose respective co-ordinates are:*

$$\begin{aligned}&\textit{Point } (1),\ (x_1,\ 0,\ 0),\\ &\textit{Point } (2),\ (0,\ x_2,\ 0),\\ &\textit{Point } (3),\ (0,\ 0,\ x_3),\end{aligned}$$

are the three vertices of the fundamental triangle, no matter what the absolute values of x_1, x_2, *and* x_3.

NOTE. The introduction of the factors μ being a mere matter of convenience, the student may ordinarily leave them out of consideration, which is the same as to suppose them unity. Their introduction amounts to supposing that the distances from the different sides of the triangle may be expressed in three different units of length without destroying the truth of our conclusions.

EXERCISES.

Prove:

1. The distances of a point from the three sides of the fundamental triangle cannot all be negative.

2. Assuming the factors μ_1, μ_2 and μ_3 to all have the same sign, every point whose co-ordinates are all positive or all negative lies within the fundamental triangle.

3. If the factors μ are all positive and the trilinear co-ordinates of a point all negative, the value of ρ must be negative.

4. The lines from the point (x_1, x_2, x_3) to the three vertices of the triangle divide the internal angles in the respective sine-ratios

$$-\frac{\mu_2 x_1}{\mu_1 x_2}, \qquad -\frac{\mu_3 x_2}{\mu_2 x_3}, \qquad -\frac{\mu_1 x_3}{\mu_3 x_1}.$$

319. *Relation of Trilinear and Rectangular Co-ordinates.* Let us suppose the three sides of the fundamental triangle to be given by their general equations in rectangular co-ordinates, as follows:

$$\left.\begin{aligned}&\text{Side 1, } ax + by + c = 0;\\ &\text{Side 2, } a'x + b'y + c' = 0;\\ &\text{Side 3, } a''x + b''y + c'' = 0.\end{aligned}\right\} \qquad (7)$$

Then, by §§ 41, 54, if x and y be the rectangular co-ordinates of any point whatever, the expression

$$ax + by + c \qquad (a)$$

will represent the distance of that point from the side (1) of the triangle, multiplied by the factor $\sqrt{a^2 + b^2}$. Since, by multiplying the equation of side (1) by an appropriate factor, we can give the quantity $\sqrt{a^2 + b^2}$ any value we please, we can make it equal to μ_1. We shall then have, when x and y are the rectangular co-ordinates of a point distant d_1 from the side (1),

$$\mu_1 d_1 = ax + by + c.$$

Thus the equations (6) of § 317 may be replaced by

$$\left.\begin{array}{l} ax \ \ + by \ \ + c \ \ = \rho x_1; \\ a'x \ + b'y \ + c' \ = \rho x_2; \\ a''x + b''y + c'' = \rho x_3. \end{array}\right\} \qquad (8)$$

These equations determine ρ, x and y when x_1, x_2 and x_3 are given. The values of x and y thus obtained from them are

$$\left.\begin{array}{l} x = \dfrac{(b'c'' - b''c')x_1 + (b''c - bc'')x_2 + (bc' - b'c)x_3}{(a'b'' - a''b')x_1 + (a''b - ab'')x_2 + (ab' - a'b)x_3}; \\ y = \dfrac{(a''c' - a'c'')x_1 + (ac'' - a''c)x_2 + (a'c - ac')x_3}{(a'b'' - a''b')x_1 + (a''b - ab'')x_2 + (ab' - a'b)x_3}; \end{array}\right\} \qquad (9)$$

which are the expressions for the rectangular co-ordinates of a point $(x. y)$ in terms of its trilinear co-ordinates.

320. *Equation of a Straight Line in Trilinear Co-ordinates.*

THEOREM. *If the trilinear co-ordinates of a point are required to satisfy a linear equation, the locus of the point will be a straight line.*

Proof. Let

$$p_1 x_1 + p_2 x_2 + p_3 x_3 = 0$$

be the linear equation which the co-ordinates must satisfy. If we substitute for x_1, x_2 and x_3 their expressions (8) in terms of Cartesian co-ordinates, we readily see that the equation

will be of the first degree in x and y. It is therefore the equation of a straight line.

321. *Homogeneous Character of Equations.* In order that any equation in trilinear co-ordinates may represent a locus, the equation must be homogeneous in terms of such co-ordinates. For, by definition, the position of a point remains unaltered when its three co-ordinates are all multiplied by any arbitrary common factor ρ. When we take a homogeneous equation of the nth degree in x_1, x_2 and x_3, such, for example, as (when $n = 2$)

$$ax_1x_2 + bx_3^2 = 0,$$

and multiply x_1, x_2 and x_3 by ρ, the result is the same as if we multiplied each member of the equation by ρ^n. Hence the relation between x_1, x_2 and x_3 expressed by the equation remains unaltered. But if we take such an equation as

$$ax_1x_2 + bx_3 + c = 0,$$

and multiply x_1, x_2 and x_3 by c, the result is

$$a\rho^2x_1x_2 + b\rho x_3 + c = 0,$$

an equation which expresses a different relation from the other. Hence such an equation cannot represent a definite locus so long as the trilinear co-ordinates are used to correspond to their definition.

Correlative of Trilinear Co-ordinates.— Co-ordinates of a Line.

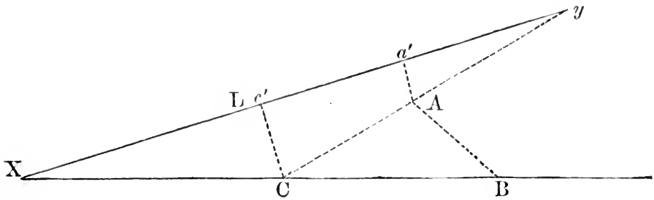
322. The principle of duality is applicable to all the preceding propositions which express position. We shall therefore change these propositions into their correlatives.

THEOREM. *If three fixed points not in the same straight line be given, the position of a line is completely determined when the mutual ratios of its distances from these points are given.*

Proof. Let the three fixed points be A, B and C.

Let it be given that the ratio of the distances of a line from the points A and C is $m : n$.

Let L be any line fulfilling this condition, and let y be the point in which it cuts the line AC. Also let Aa' and Cc'



be the perpendiculars from A and C upon the line L. We then have, by hypothesis,

$$Aa' : Cc' = m : n.$$

Hence, by similar triangles,

$$Ay : Cy = m : n.$$

This last relation completely fixes the position of the point y (§ 279), which therefore remains the same for all lines which satisfy the given condition. That is,

Every line fulfilling the condition that its distances from two fixed points, A and C, shall be in the ratio $m : n$, passes through that point which divides the junction-line AC in the ratio $m : n$.

Let it also be given that the distances of the line from the points C and B shall be in the ratio $n : p$. It must then pass through a point X which divides the junction-line CB in the ratio $n : p$.

If now it be required that the line shall satisfy both conditions, it must pass through both the points X and y, and is therefore completely fixed.

When both these conditions are satisfied, the distances of the line from BA will be in the ratio $p : m$, and it will intersect AB in some point, dividing AB in the ratio $p : m$.

The three points then satisfy the proposition of (§ 287), because

$$\frac{m}{n} \cdot \frac{n}{p} \cdot \frac{p}{m} = 1.$$

Proceeding as in the case of the point, if we put

D_1, D_2, D_3, the distances of the line from the three points;

λ_1, λ_2, λ_3, constant factors,

we may express the mutual ratios of the distances by three quantities, u_1, u_2 and u_3, proportional to them. We then have

$$u_1 : u_2 : u_3 = \lambda_1 D_1 : \lambda_2 D_2 : \lambda_3 D_3;$$
$$\frac{\lambda_1 D_1}{u_1} = \frac{\lambda_2 D_2}{u_2} = \frac{\lambda_3 D_3}{u_3} \equiv \sigma;$$
$$\lambda_1 D_1 = \sigma u_1;$$
$$\lambda_2 D_2 = \sigma u_2;$$
$$\lambda_3 D_3 = \sigma u_3.$$

Thus the position of the line is completely fixed by the three quantities u_1, u_2 and u_3, which are therefore called *co-ordinates* of the line. We therefore have the definition:

The **trilinear co-ordinates** of a line are three numbers proportional to its distances from three fixed points, each distance being first multiplied by any fixed factor.

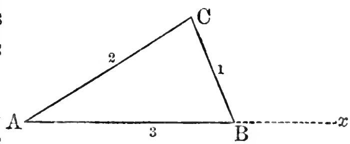
The junction-lines of the three points form the **fundamental triangle** of reference.

323. There are therefore two ways of defining the position of a line, namely:

1. By the equation of the line.
2. By the value of its co-ordinates.

To investigate the relation of these two ways let us consider the problem: What are the co-ordinates of the line whose equation is

$$mx_1 + nx_2 + px_3 = 0? \qquad (a)$$



Let us suppose the indices 1, 2 and 3 to refer to the sides BC, CA and AB respectively. Let us first find the point x in which the line (a) intersects AB. To do this we put $x_3 = 0$, which gives

$$\frac{x_1}{x_2} = -\frac{n}{m};$$

and by substituting for x_1 and x_2 their expressions (6) in terms of the distances,

$$\frac{d_1}{d_2} = -\frac{\mu_2 n}{\mu_1 m}.$$

Since d_1 and d_2 are the distances of x from the sides CA and CB, $-\frac{d_1}{d_2}$ is the sine-ratio in which the line Cx divides the angle C (§ 281). To define the point x by the distance-ratio in which it cuts AB, we have the equations

$$\frac{Bx}{Cx} = \frac{\sin BCx}{\sin B};$$

$$\frac{Ax}{Cx} = \frac{\sin ACx}{\sin A}.$$

Hence

$$\frac{Bx}{Ax} = \frac{\sin BCx}{\sin ACx} \cdot \frac{\sin A}{\sin B} = -\frac{d_1 \sin A}{d_2 \sin B} = \frac{\mu_2 n \sin A}{\mu_1 m \sin B}, \qquad (b)$$

which determines the point of intersection, x.

By what has been shown, this distance-ratio is the ratio of the two co-ordinates u_1 and u_2 of the line, multiplied by a factor. In fact, putting D_1 and D_2 for the distances of the line from A and B respectively, we have

$$\frac{Bx}{Ax} = \frac{D_2}{D_1} = \frac{\lambda_1 u_2}{\lambda_2 u_1}.$$

Comparing this with the equation (b), we have, for the ratio of the two line-coördinates,

$$\frac{u_2}{u_1} = \frac{\lambda_2 \mu_2 n \sin A}{\lambda_1 \mu_1 m \sin B} = \frac{n}{m} \cdot \frac{\lambda_2 \mu_2 \sin A}{\lambda_1 \mu_1 \sin B}.$$

In the same way,

$$\frac{u_3}{u_2} = \frac{\lambda_3 \mu_3 p \sin B}{\lambda_2 \mu_2 n \sin C} = \frac{p}{n} \cdot \frac{\lambda_3 \mu_3 \sin B}{\lambda_2 \mu_2 \sin C}; \qquad (c)$$

$$\frac{\mu_1}{\mu_3} = \frac{\lambda_1 \mu_1 m \sin C}{\lambda_3 \mu_3 p \sin A} = \frac{m}{p} \cdot \frac{\lambda_1 \mu_1 \sin C}{\lambda_3 \mu_3 \sin A}.$$

Now, it will be remembered that the constant factors λ and μ which multiply the distances between the points and

lines have been left entirely arbitrary. Can we not so determine them that the last fractions in the third members of the above equations shall be unity? This will require the factors to fulfil the conditions

$$\left.\begin{aligned} \lambda_1\mu_1 \sin B &= \lambda_2\mu_2 \sin A; \\ \lambda_2\mu_2 \sin C &= \lambda_3\mu_3 \sin B; \\ \lambda_3\mu_3 \sin A &= \lambda_1\mu_1 \sin C. \end{aligned}\right\} \qquad (d)$$

These three equations are really equivalent to but two, because any one can be deduced from the other two by eliminating the common angle. If we suppose the values of μ to be given, we can determine the mutual ratios of the λ's by the equations

$$\frac{\lambda_1}{\lambda_3} = \frac{\mu_3 \sin A}{\mu_1 \sin C};$$
$$\frac{\lambda_2}{\lambda_3} = \frac{\mu_3 \sin B}{\mu_2 \sin C}.$$

Hence the required condition can always be satisfied, and we shall always suppose it satisfied.

The equations (c) can then be satisfied by putting

$$\begin{aligned} u_1 &= m \times \text{any factor}; \\ u_2 &= n \times \text{the same factor}; \\ u_3 &= p \times \text{the same factor}. \end{aligned}$$

Hence:

THEOREM. *If the co-ordinates x_1, x_2, x_3 of a point are considered as variables required to satisfy the equation*

$$mx_1 + nx_2 + px_3 = 0,$$

the point will always lie on the line whose co-ordinates are the constants m, n and p, or their multiples.

324. *Equation of a Point.* We have the following theorem, the correlative of the preceding one.

THEOREM. *All lines whose co-ordinates u_1, u_2 and u_3 satisfy a linear equation*

$$mu_1 + nu_2 + pu_3 = 0$$

pass through a point, namely, the point whose co-ordinates are m, n and p.

This theorem follows immediately from the preceding one, because when a point lies on a certain line, the line passes through that point. Putting both equations in the form

$$u_1x_1 + u_2x_2 + u_3x_3 = 0,$$

the theorem of § 323 asserts that whenever this equation is satisfied the point (x_1, x_2, x_3) lies upon the line (u_1, u_2, u_3). Changing the form but not the essence of the conclusion, we have the theorem that whenever this equation is satisfied the line (u_1, u_2, u_3) passes through the point (x_1, x_2, x_3). This result, being true for all values of the six quantities which satisfy the equation, will remain true when we suppose x_1, x_2 and x_3 to remain constant and u_1, u_2 and u_3 to vary; that is, the varying line (u_1, u_2, u_3) will then constantly pass through the fixed point (x_1, x_2, x_3).

325. The preceding conclusions may be summed up as follows:

I. *If u_1, u_2 and u_3 are line-coördinates, and x_1, x_2 and x_3 are point-coördinates, then, so long as these co-ordinates are unrestricted, they may represent any line and any point whatever.*

II. *If it be required that the line shall pass through the point and the point lie on the line, the co-ordinates must satisfy the condition*

$$u_1x_1 + u_2x_2 + u_3x_3 = 0.$$

III. *If, in this equation, we suppose the x's to remain fixed while the u's vary, the lines represented by the u's will all pass through the fixed point represented by the x's.*

IV. *If we suppose the x's to vary while the u's remain constant, the points represented by the x's will all lie on the fixed line represented by the u's.*

326. For brevity of writing we may use a single letter to represent the combination of three co-ordinates of a point or line. Then the expression "the point (p)" will mean the point whose co-ordinates are p_1, p_2 and p_3.

THEOREM. *If (x) and (y) are any two points, and if, with any factor λ, we form the quantities*

$$\left.\begin{aligned} z_1 &= x_1 + \lambda y_1, \\ z_2 &= x_2 + \lambda y_2, \\ z_3 &= x_3 + \lambda y_3, \end{aligned}\right\} \qquad (a)$$

the point (z) will lie on the line joining the points (x) and (y).

Proof. Let (p) be the line joining the points (x) and (y). Because the line (p) passes through the point (x), we have

$$p_1x_1 + p_2x_2 + p_3x_3 = 0.$$

Because (p) passes through the point (y), we also have

$$p_1y_1 + p_2y_2 + p_3y_3 = 0.$$

Multiplying this equation by λ and adding it to the other,

$$p_1(x_1 + \lambda y_1) + p_2(x_2 + \lambda y_2) + p_3(x_3 + \lambda y_3) = 0,$$

or
$$p_1z_1 + p_2z_2 + p_3z_3 = 0.$$

Therefore the point (z) is on the line (p). Q. E. D.

Cor. Any point (z) whose co-ordinates satisfy the conditions

$$\begin{aligned} z_1 &= \lambda x_1 + \mu y_1, \\ z_2 &= \lambda x_2 + \mu y_2, \\ z_3 &= \lambda x_3 + \mu y_3, \end{aligned}$$

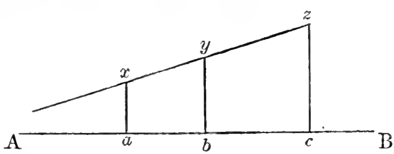
lies on the same straight line with the points (x) and (y), whatever be the factors λ and μ.*

327. PROBLEM. *Having the four points in a right line,*

$$(x), \qquad (y), \qquad (x + \lambda y), \qquad (x + \lambda' y),$$

it is required to determine their anharmonic ratio.

We must first, instead of the trilinear co-ordinates, take the actual reduced distances of the points from the sides of the triangle. Let us then suppose



* These coefficients λ and μ must, of course, not be confounded with the factors used in §§ 317, 322.

AB, any side of the fundamental triangle;
$z \equiv$ the point $(x+\lambda y)$;
ax, by, $cz \equiv p$, q and r, the distances of x, y and z from AB.

We then have for the distance-ratio of z, with respect to x and y,

$$\frac{xz}{yz} = \frac{r-p}{r-q} \equiv \mu.$$

This gives

$$r = \frac{p - \mu q}{1 - \mu}.$$

We have three equations of this form, corresponding to the three sides of the fundamental triangle, in which p, q and r have the respective indices 1, 2 and 3. We may write them:

$$(1-\mu)r_1 = p_1 - \mu q_1;$$
$$(1-\mu)r_2 = p_2 - \mu q_2;$$
$$(1-\mu)r_3 = p_3 - \mu q_3.$$

The trilinear co-ordinates x_1, x_2 and x_3, proportional to p_1, p_2 and p_3, are formed by dividing these last quantities by a common factor $\equiv \rho$. Let σ be the common factor for q. Then these equations become, by substitution and reduction,

$$\frac{1-\mu}{\rho} r_1 = x_1 - \frac{\mu\sigma}{\rho} y_1;$$
$$\frac{1-\mu}{\rho} r_2 = x_2 - \frac{\mu\sigma}{\rho} y_2;$$
$$\frac{1-\mu}{\rho} r_3 = x_3 - \frac{\mu\sigma}{\rho} y_3.$$

These equations become identical with (a) by putting

$$z = \frac{1-\mu}{\rho} r; \quad -\frac{\mu\sigma}{\rho} = \lambda, \qquad \text{or} \qquad \mu = -\frac{\lambda\rho}{\sigma}.$$

But μ is the distance-ratio of the point $(z) \equiv (x+\lambda y)$ with respect to the points (x) and (y). Hence,

When from two points, (x) and (y), we form the third, $(x+\lambda y)$, in the same straight line, the distance-ratio of the

third with respect to the other two is equal to λ *multiplied by a factor,* $-\frac{\rho}{\sigma}$, *depending upon the absolute values of the trilinear co-ordinates.*

Since, from the nature of trilinear co-ordinates, this factor remains indeterminate, the distance-ratio also remains indeterminate. But if we also take the distance-ratio of a fourth point, $(x + \lambda' y)$, and then form the anharmonic ratio, this factor will divide out, and we shall have

$$\text{Anh. ratio} = \frac{\lambda}{\lambda'}.$$

Hence the anharmonic ratio of the four points (x), (y), $(x + \lambda y)$, $(x + \lambda' y)$ is $\frac{\lambda}{\lambda'}$, which solves the problem.

Cor. Harmonic Points. Since four harmonic points are such whose anharmonic ratio is -1, we must then have $\lambda' = -\lambda$. We therefore conclude that if we have any four points capable of being expressed in the form

$$(x), \qquad (y), \qquad (x + \lambda y), \qquad (x - \lambda y),$$

the last pair will divide harmonically the segment contained by the first pair.

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The following classified list of books is prepared for the use of students who wish to continue the study of the subject:

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CHASLES, *Traité de Géométrie Supérieure* (546 pages 8vo), is noted for its elegance of treatment. It is principally confined to the geometry of lines and circles. The subject is continued in

CHASLES, *Traité des Sections Coniques, Première Partie* (no second part published).

TOWNSEND, *Modern Geometry of the Point, Line and Circle* (2 vols.; Dublin, 1863), covers ground similar to the first work of CHASLES, but is more elementary.

STEINER, *Vorlesungen über Synthetische Geometrie*, is a very extended treatise, but lacks the clear presentation of CHASLES.

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